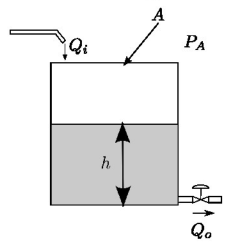
Universidad Pontificia Bolivariana
Facultad de Ingeniería Aeronáutica
Automatic Flight Control
Problem set 2
Mathematical Modeling of Dynamical Systems

1. Consider a cylindrical tank as shown in the figure.



Where:

Q_i : input flow $[=]m^3/s$.

Q_o : output flow through the valve (output of the tank) $[=]m^3/s$.

h : tank level $[=]m$.

A : tank cross-section area $[=]m^2$.

P_A : atmospheric pressure $[=]Pa$.

ρ : density of the liquid $[=]kg/m^3$.

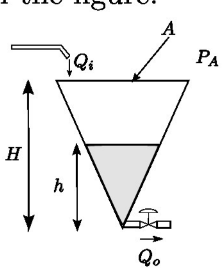
g : acceleration of gravity $[=]m/s^2$.

Consider that the valve characteristic is given by $Q_o = K\sqrt{\Delta P}$, where ΔP is the pressure difference between the valve terminals and K is a given constant.

(a) Obtain the mathematical model of this system in state space considering Q_i as input and Q_o as output.

(b) Obtain the mathematical model of this system in terms of a differential equation, considering Q_i as input and h as output.

2. Consider a conical tank as shown in the figure.



Where:

Q_i : input flow $[=] m^3/s$.

Q_o : output flow through the valve (output of the tank) $[=] m^3/s$.

h : tank level $[=] m$.

A : cross-sectional area of the liquid surface in the tank when the level is maximum ($h = H$) $[=] m^2$.

H : the maximum tank level $[=] m$.

P_A : atmospheric pressure $[=] Pa$.

ρ : density of the liquid $[=] kg/m^3$.

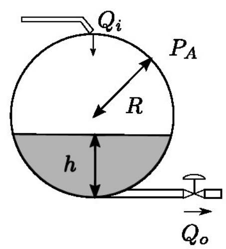
g : acceleration of gravity $[=] m/s^2$.

Consider that the valve characteristic is given by $Q_o = K\sqrt{\Delta P}$, where ΔP is the pressure difference between the valve terminals and K is a given constant.

(a) Obtain the mathematical model of this system in state space considering Q_i as input and Q_o as output.

(b) Obtain the mathematical model of this system in terms of a differential equation, considering Q_i as input and h as output.

3. Consider a spherical tank as shown in the figure.



Where:

Q_i : input flow $[=] m^3/s$.

Q_o : output flow through the valve (output of the tank) $[=] m^3/s$.

h : tank level $[=] m$.

R : radius of the spherical tank $[=] m$.

P_A : atmospheric pressure $[=] Pa$.

ρ : density of the liquid $[=] kg/m^3$.

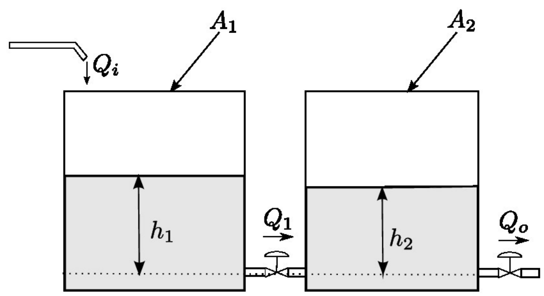
g : acceleration of gravity $[=] m/s^2$.

Consider that the valve characteristic is given by $Q_o = K\sqrt{\Delta P}$, where ΔP is the pressure difference between the valve terminals and K is a given constant.

(a) Obtain the mathematical model of this system in state space considering Q_i as input and Q_o as output.

(b) Obtain the mathematical model of this system in terms of a differential equation, considering Q_i as input and h as output.

4. Consider the **communicated** tanks system shown in the figure.



Where:

Q_i : input flow (input signal) $[=] m^3/s$.

Q_1 : flow through valve 1 (between tank 1 and tank 2) $[=] m^3/s$.

Q_o : output flow through valve 2 (output signal) $[=] m^3/s$.

h_1 : tank 1 level $[=] m$.

h_2 : tank 2 level $[=] m$.

A_1 : tank 1 cross-sectional area $[=] m^2$.

A_2 : tank 2 cross-sectional area $[=] m^2$.

P_A : atmospheric pressure $[=] Pa$.

ρ : density of the liquid $[=] kg/m^3$.

g : acceleration of gravity $[=] m/s^2$.

Consider that the characteristic of each valve is linear, so that $Q = K\Delta P$, where Q is the flow rate of the valve, ΔP is the pressure difference between the valve terminals and K is a given constant.

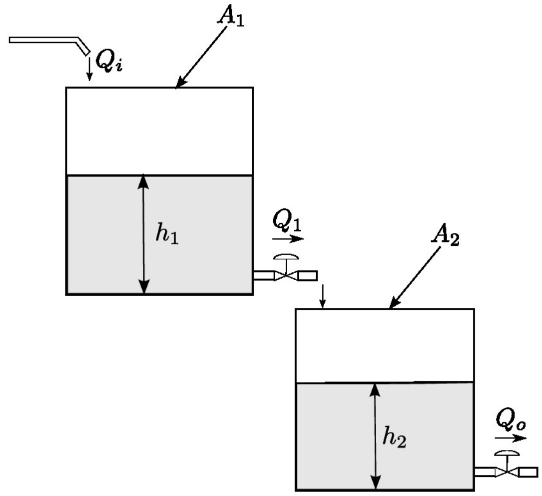
(a) Obtain the mathematical model of this system in state space considering Q_i as input and Q_1 and Q_o as outputs.

(b) Obtain the mathematical model of this system in terms of a differential equation, considering Q_i as input and Q_o as output.

(c) Obtain the mathematical model of this system in terms of its transfer function, considering Q_i as input and Q_o as output.

5. Obtain the mathematical model in state space of the system of the previous problem assuming that the characteristic of each valve is given by $Q = K\sqrt{|\Delta P|}Sgn\,(\Delta P)$, where Q is the valve flow, ΔP is the pressure difference between the valve terminals and K is a given constant.

6. Consider the cascade tank system shown in the figure.



Where:

Q_i : input flow (input signal) $[=]m^3/s$.

Q_1 : flow through valve 1 (between tank 1 and tank 2) $[=]m^3/s$.

Q_o : output flow through valve 2 (output signal) $[=]m^3/s$.

h_1 : tank 1 level $[=]m$.

h_2 : tank 2 level $[=]m$.

A_1 : tank 1 cross-sectional area $[=]m^2$.

A_2 : tank 2 cross-sectional area $[=]m^2$.

P_A : atmospheric pressure $[=]Pa$.

ρ : density of the liquid $[=]kg/m^3$.

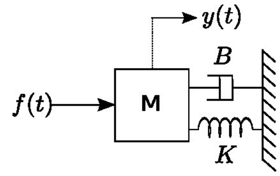
g : acceleration of gravity $[=]m/s^2$.

Consider that the characteristic of each valve is given by $Q = K\sqrt{|\Delta P|}Sgn\,(\Delta P)$, where Q is the valve flow, ΔP is the pressure difference between the valve terminals and K is a given constant.

(a) Obtain the mathematical model of this system in state space considering Q_i as input and Q_1 and Q_o as outputs.

(b) Obtain the mathematical model of this system in terms of a differential equation, considering Q_i as input and Q_o as output.

7. Consider the mass-spring-damper system in the figure, where $f(t)$ which is the force applied to the mass corresponds to the input signal, and $y(t)$ which is the displacement of the mass corresponds to the output signal.



Where:

f : external force applied to the mass $[=]N$.

y : position of the mass with respect to an inertial reference frame $[=]m$.

M : mass $[=]kg$.

K : spring constant $[=]kg/s^2$.

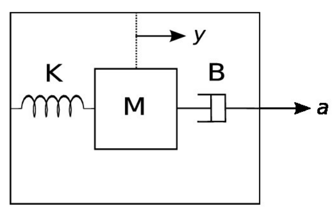
B : viscous friction constant $[=]kg/s$.

(a) Obtain the mathematical model in terms of a differential equation.

(b) Obtain the mathematical model in terms of the transfer function of the system.

(c) If the state variables $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$ are defined, obtain the mathematical model in state space.

8. Obtain the mathematical model of the accelerometer shown in the figure, where the acceleration of the accelerometer, a, is the input and the displacement of the mass with respect to the body of the accelerometer, y, is the output.



Where:

a : accelerometer acceleration respect to a inertial frame $[=]m/s^2$.

y : position of the test mass in the accelerometer $[=]m$.

M : test mass of the accelerometer $[=]kg$.

K : spring constant $[=]kg/s^2$.

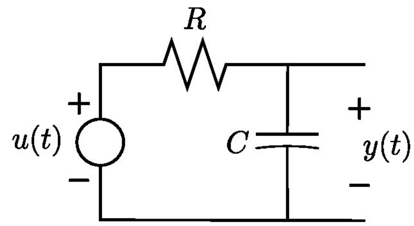
B : viscous friction constant $[=]kg/s$.

(a) Obtain the mathematical model in terms of a differential equation.

(b) Obtain the mathematical model in terms of the transfer function of the system.

(c) If the state variables $x_1(t) = \dot{y}(t)$ and $x_2(t) = y(t)$ are defined, obtain the mathematical model in state space.

9. Consider an RC electrical circuit as shown in the figure.

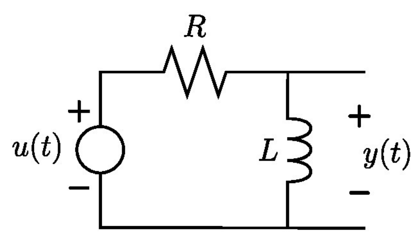


Where:

$u(t)$: source voltage [=]V.
$y(t)$: output voltage (voltage across C) [=]V.
R : electrical resistance [=]Ω.
C : capacitance [=]F.

(a) Obtain the mathematical model in terms of a differential equation.
(b) Obtain the mathematical model in terms of the transfer function of the system.
(c) Obtain the mathematical model in state space.

10. Consider an RL electrical circuit as shown in the figure.

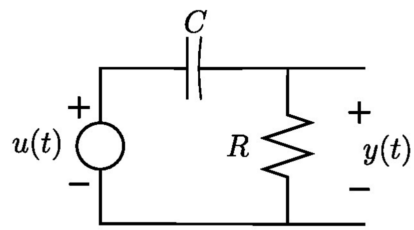


Where:

$u(t)$: source voltage [=]V.
$y(t)$: output voltage (voltage across L) [=]V.
R : electrical resistance [=]Ω.
L : inductance [=]H.

(a) Obtain the mathematical model in terms of a differential equation.
(b) Obtain the mathematical model in terms of the transfer function of the system.
(c) Obtain the mathematical model in state space.

11. Consider an CR electrical circuit as shown in the figure.

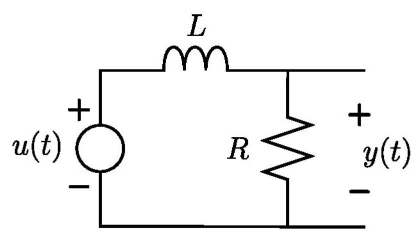


Where:

$u(t)$: source voltage [=]V.
$y(t)$: output voltage (voltage across R) [=]V.
R : electrical resistance [=]Ω.
C : capacitance [=]F.

(a) Obtain the mathematical model in terms of a differential equation.
(b) Obtain the mathematical model in terms of the transfer function of the system.
(c) Obtain the mathematical model in state space.

12. Consider an LR electrical circuit as shown in the figure.

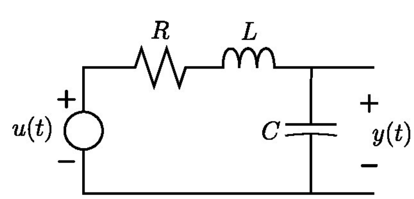


Where:

$u(t)$: source voltage [=]V.
$y(t)$: output voltage (voltage across R) [=]V.
R : electrical resistance [=]Ω.
L : inductance [=]H.

(a) Obtain the mathematical model in terms of a differential equation.
(b) Obtain the mathematical model in terms of the transfer function of the system.
(c) Obtain the mathematical model in state space.

13. Consider an RLC electrical circuit as shown in the figure.

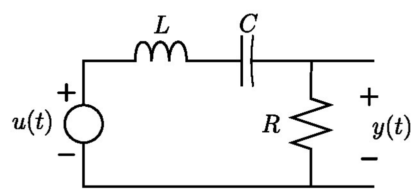


Where:

$u(t)$: source voltage $[=]V$.
$y(t)$: output voltage (voltage across C) $[=]V$.
R : electrical resistance $[=]\Omega$.
L : inductance $[=]H$.
C : capacitance $[=]F$.

(a) Obtain the mathematical model in terms of a differential equation.
(b) Obtain the mathematical model in terms of the transfer function of the system.
(c) Obtain the mathematical model in state space.

14. Consider an LCR electrical circuit as shown in the figure.

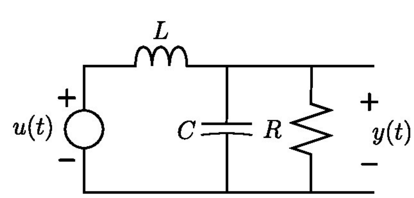


Where:

$u(t)$: source voltage $[=]V$.
$y(t)$: output voltage (voltage across R) $[=]V$.
R : electrical resistance $[=]\Omega$.
L : inductance $[=]H$.
C : capacitance $[=]F$.

(a) Obtain the mathematical model in terms of a differential equation.
(b) Obtain the mathematical model in terms of the transfer function of the system.
(c) Obtain the mathematical model in state space.

15. Consider an LCR electrical circuit as shown in the figure.



Where:

$u(t)$: source voltage $[=]V$.
$y(t)$: output voltage (voltage across C or R) $[=]V$.
R : electrical resistance $[=]\Omega$.

L : inductance $[=]H$.

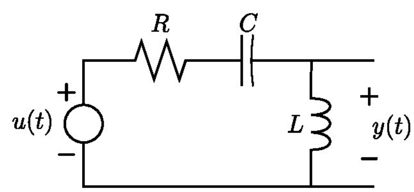
C : capacitance $[=]F$.

(a) Obtain the mathematical model in terms of a differential equation.

(b) Obtain the mathematical model in terms of the transfer function of the system.

(c) Obtain the mathematical model in state space.

16. Consider an RCL electrical circuit as shown in the figure.



Where:

$u(t)$: source voltage $[=]V$.

$y(t)$: output voltage (voltage across L) $[=]V$.

R : electrical resistance $[=]\Omega$.

L : inductance $[=]H$.

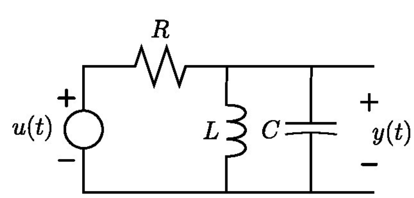
C : capacitance $[=]F$.

(a) Obtain the mathematical model in terms of a differential equation.

(b) Obtain the mathematical model in terms of the transfer function of the system.

(c) Obtain the mathematical model in state space.

17. Consider an electrical RLC circuit with a parallel LC tank circuit as shown in the figure.



Where:

$u(t)$: source voltage $[=]V$.

$y(t)$: output voltage (voltage across L or C) $[=]V$.

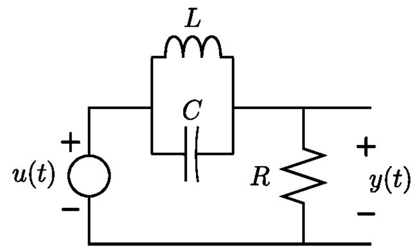
R : electrical resistance $[=]\Omega$.

L : inductance $[=]H$.

C : capacitance $[=]F$.

(a) Obtain the mathematical model in terms of a differential equation.
(b) Obtain the mathematical model in terms of the transfer function of the system.
(c) Obtain the mathematical model in state space.

18. Consider an RLC electrical circuit with a series LC tank circuit as shown in the figure.

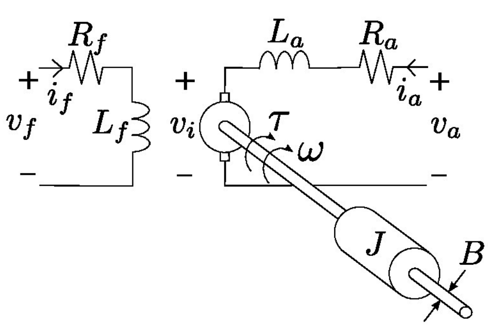


Where:

$u(t)$: source voltage $[=]V$.
$y(t)$: output voltage (voltage across R) $[=]V$.
R : electrical resistance $[=]\Omega$.
L : inductance $[=]H$.
C : capacitance $[=]F$.

(a) Obtain the mathematical model in terms of a differential equation.
(b) Obtain the mathematical model in terms of the transfer function of the system.
(c) Obtain the mathematical model in state space.

19. Consider a DC motor as shown in the figure.



Where:

v_f : voltage applied to field winding $[=]V$.
i_f : current through field winding $[=]A$.
v_a : voltage across armature winding $[=]V$.
i_a : current through armature winding $[=]A$.

R_f : field winding resistance $[=]\Omega$.

L_f : field winding inductance $[=]H$.

R_a : armature winding resistance $[=]\Omega$.

L_a : armature winding inductance $[=]H$.

ω : angular velocity in motor axis $[=]rad/s$.

J : moment of inertia of rotating parts of the motor $[=]kg.m^2$.

B : viscous friction coefficient of rotating parts of the motor $[=]kg.m^2/s$.

$v_i = Ki_f\omega$: voltage induced in the armature $[=]V$. K is a constant associated with the motor.

$\tau = K\eta i_f i_a$: torque produced by the motor $[=]N \cdot m$. η is the efficiency of the motor $(0 \leq \eta \leq 1)$.

(a) Obtain the mathematical model in state space assuming that the voltage of the field winding, v_f, and the armature winding voltage, v_a, are the inputs, and that the angular velocity on the motor shaft, ω, is the output.

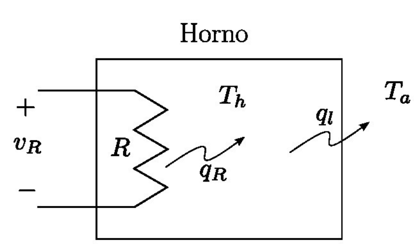
(b) Obtain the mathematical model in terms of a differential equation assuming that the field winding voltage, v_f, is the input, that the winding current of the armature, i_a, is a constant parameter I_a, and that the angular velocity on the axis of the motor, ω, is the output.

(c) Obtain the mathematical model in transfer function assuming that the voltage of the field winding, v_f, is the input, that the armature winding current, i_a, is a constant parameter I_a, and that the angular speed on the motor axis, ω, is the output.

(d) Obtain the mathematical model in terms of a differential equation assuming that the armature winding voltage, v_a, is the input, the field winding voltage, v_f, is a constant parameter V_f, and that the angular velocity on the motor axis, ω, is the output.

(e) Obtain the mathematical model in transfer function assuming that the voltage of the armature winding, v_a, is the input, the field winding voltage, v_f, is a constant parameter V_f, and that the angular velocity on the motor axis, ω, is the output.

20. Consider a small electrical oven as shown in the figure.



Where:

T_a : room temperature $[=]^\circ C$.

T_h : temperature inside the oven, it is assumed uniform $[=]^\circ C$.

R_θ : thermal resistance from inside the oven to the environment $[=]°C \cdot s/kcal$.

C_θ : thermal capacitance of the oven interior $[=]kcal/°C$.

P_R : electric power dissipated in the oven resistance $[=]W$.

R : electric resistance of the oven resistance $[=]\Omega$.

v_R : voltage applied to the oven resistance $[=]V$.

Note that:

$P_R = \frac{v_R^2}{R}$

$q_R = KP_R$ where q_R is the heat flux generated by the electrical resistance and K is a constant associated with the oven.

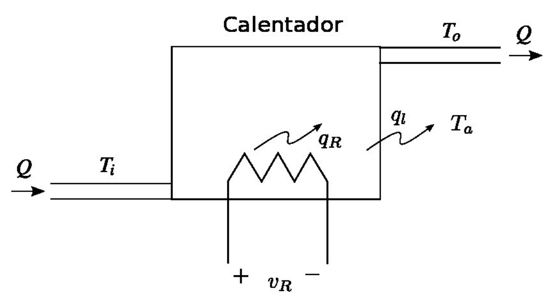
$q_l = \frac{T_h - T_a}{R_\theta}$ where q_l is the heat flow lost in the environment.

$q_h = C_\theta \frac{dT_h}{dt}$ where q_h is the heat flow stored in the oven..

(a) Obtain a state space model of this system.

(b) Obtain the differential equation that models this system considering v_R as the input and T_h as the output.

21. Consider an electric water heater as shown in the figure.



Where:

Q : flow of water $[=]m^3/s$.

T_a : room temperature $[=]°C$.

T_i : input water temperature $[=]°C$.

T_o : output water temperature $[=]°C$.

R_θ : thermal resistance of the heater insulation (from inside the heater to the environment) $[=]°C \cdot s/kcal$.

C_θ : thermal capacitance inside the heater $[=]kcal/°C$.

V : heater volume $[=]m^3$.

ρ : water density $[=]kg/m^3$.

c : specific heat of water $[=]kcal/(kg \cdot° C)$.

P_R : electrical power dissipated in the resistance of the heater $[=]W$.

R : electrical resistance of the heater resistance $[=]\Omega$.

v_R : voltage applied to the heater resistance $[=]V$.

Note that:

$P_R = \frac{v_R^2}{R}$

$q_R = KP_R$ where q_R is the heat flow generated by the electrical resistance and K is a constant associated with the heater.

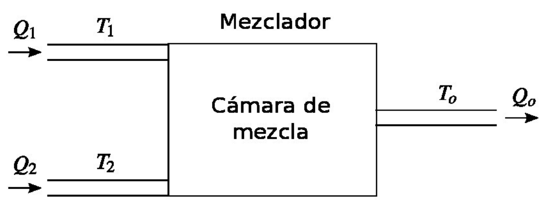
$q_l = \frac{T_o - T_a}{R_\theta}$ where q_l is the heat flow lost in the environment.

$q_c = C_\theta \frac{dT_o}{dt}$ where q_c is the heat flow stored in the heater with $C_\theta = c\rho V$.

(a) Obtain a state space model of this system assuming that Q, T_i, and v_R are the inputs and T_o is the output.

(b) Obtain the differential equation that models this system considering v_R as the input, assuming that Q, T_i are constant and considering T_o as the output.

22. Consider a water mixer as shown in the figure.



Where:

Q_1 : volumetric flow of cold water $[=]m^3/s$.

T_1 : temperature of input cold water $[=]°C$.

Q_2 : volumetric flow of hot water $[=]m^3/s$.

T_2 : temperature of input hot water $[=]°C$.

Q_o : volumetric flow of the mixer output water $[=]m^3/s$.

T_o : mixer output water temperature $[=]°C$.

C_θ : thermal capacitance of mixer mixing chamber $[=]kcal/°C$.

V : volume of mixer mixing chamber $[=]m^3$.

ρ : water density $[=]kg/m^3$.

c : specific heat of water $[=]kcal/(kg \cdot° C)$.

Neglect heat losses to the environment and note that

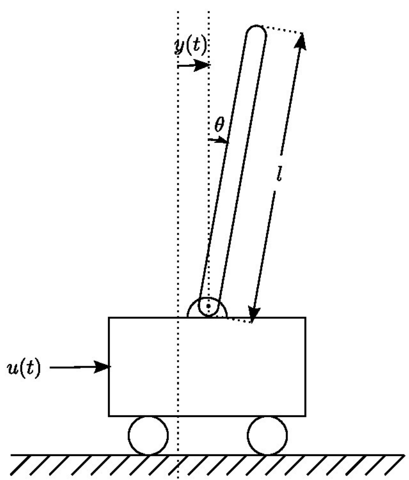
$Q_o = Q_1 + Q_2$

$q_c = C_\theta \frac{dT_o}{dt}$ where q_c is the heat flow stored in the mixer mixing chamber with $C_\theta = c\rho V$.

(a) Obtain a state space model of this system assuming that Q_1, T_1, Q_2, and T_2 are the inputs and Q_o and T_o are the outputs.

(b) Obtain the differential equation that models this system considering Q_1, T_1, Q_2, and T_2 as the inputs and considering T_o as the output. salida.

23. Consider an inverted pendulum as shown in the figure.



Where:

$u(t)$: force applied to the car $[=]N$.

$y(t)$: car displacement $[=]m$.

$\theta(t)$: angle of the pendulum respect to the vertical $[=]rad$.

M : car mass $[=]kg$.

m : pendulum mass $[=]kg$.

I : pendulum moment of inertia $[=]kg \cdot m^2$.

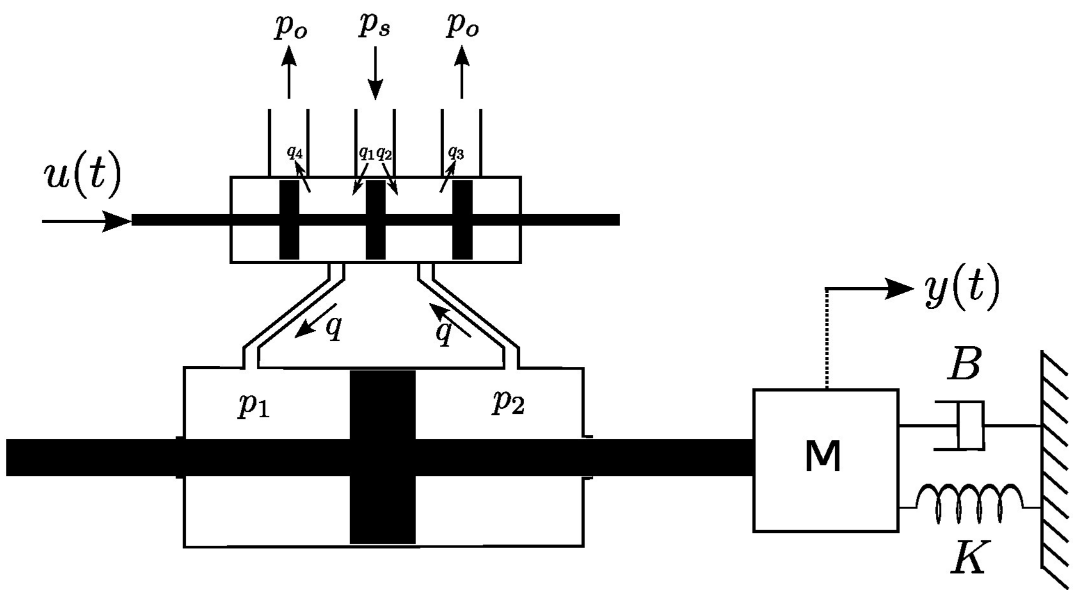
l : pendulum length $[=]m$.

B_c : car viscous friction coefficient $[=]kg/s$.

B_p : pendulum viscous friction coefficient $[=]kg \cdot m^2/s$.

Obtain a state space model of this system with $u(t)$ as the input, and with outputs $y(t)$ and $\theta(t)$.

24. Consider the hydraulic system shown in the figure.



Where:

$u(t)$: spool valve displacement $[=]N$.

$y(t)$: displacement of the mass $[=]m$.

p_s : supply pressure $[=]Pa$.

p_o : return pressure $[=]Pa$.

p_1 : pressure on the left side of the hydraulic cylinder $[=]Pa$.

p_2 : pressure on the right side of the hydraulic cylinder $[=]Pa$.

q_i : flow through the port i $[=]kg/s$.

q : flow through the hydraulic cylinder ports $[=]kg/s$.

A : hydraulic cylinder area $[=]m^2$.

ρ : hydraulic fluid density $[=]kg/m^3$.

Obtain a state space model of this system assuming that $u(t)$ is the input and $y(t)$ is the output.

25. Linearize the models of the following systems. In each case, linearize the model in state space, obtain the equations that govern the operating point, obtain the differential equation of the linearized model, and obtain the transfer function of the linearized system.

    (a) The conical tank from problem 2.
    (b) The spherical tank from problem 3.
    (c) The communicated tanks system from problem 5.
    (d) The cascade tank system from problem 6.

(e) The mass-spring-damper system from problem 7, considering a non linear spring with characteristic $f_K = Ky^3$.

(f) The DC motor from problem 19. In this case the model in state variables is multivariable with inputs v_a and v_f and output ω.

(g) Obtain the transfer function for the linearized model of the DC motor of problem 19 when v_a is a constant of value V_a.

(h) The oven from problem 20.

(i) The water heater from problem 21.

(j) The water mixer from problem 22.

(k) The inverted pendulum from problem 23.

(l) The hydraulic system of problem 24.