Automatic Flight Control

Stability, Observability, and Controllability

Luis Benigno Gutiérrez Zea

Facultad de Ingeniería Aeronáutica
Universidad Pontificia Bolivariana

First semester - 2025

Notes

Outline

Notes

Outline

Notes

Stability: the basic idea

- A system is said to be **stable** if the output or state **remains bounded or goes to zero** in some way for some conditions in the input and the initial state.
- This is not a definition of stability, for a definition of stability all the conditions must be stated.
- There are different stability criteria, each with a precise definition of stability according to that criteria.

Notes

Some stability criteria

- Stability in the sense of Lyapunov.
- Asymptotic stability.
- Exponential stability.
- BIBO stability.
- BIBS stability.

Notes

BIBO stability

- A system is **Bounded Input Bounded Output stable** (BIBO stable) if the output is bounded for any bounded input assuming zero initial state.
- That is, given that $\mathbf{u}(t)$ is the input and $\mathbf{y}(t)$ is the output of a system, such that $\mathbf{y}(t) = \mathscr{H}\left\{\mathbf{u}(t)\right\}$, where $\mathscr{H}$ is the operator that represent the system behavior, the system is BIBO stable if for any $L > 0$ there exist $M > 0$ such that if $\|\mathbf{u}(t)\| < L$ then $\|\mathbf{y}(t)\| < M$.

Notes

BIBO stability for LTI systems

- It can be shown that a linear time invariant system is **Bounded Input Bounded Output stable** (BIBO stable) if and only if the impulse response matrix is absolutely integrable, meaning that $\int_0^t \|\mathbf{h}(\lambda)\| d\lambda < M$ for some $M > 0$ and for all $t > 0$.
- Than condition is equivalent to say that **a linear time invariant system is Bounded Input Bounded Output stable (BIBO stable) if and only if all the system poles have negative real part**.

Notes

BIBS stability

- A system is **Bounded Input Bounded State stable** (BIBS stable) if the state is bounded for any bounded input and any bounded initial state.
- That is, given that $\mathbf{u}(t)$ is the input and $\mathbf{x}(t)$ is the state of a system, such that $\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$, with $\mathbf{x}(0) = \mathbf{x}_0$, the system is BIBS stable if for any $L > 0$ and $M > 0$ there exist $N > 0$ such that if $\|\mathbf{u}(t)\| < L$ and $\|\mathbf{x}_0\| < M$ then $\|\mathbf{x}(t)\| < N$ for all $t > 0$.

Notes

BIBS stability for LTI systems

- It can be shown that a linear time invariant system is **Bounded Input Bounded State stable** (BIBS stable) if and only if the state transition matrix is absolutely integrable, meaning that $\int_0^t \|\phi(\lambda)\| d\lambda < M$ for some $M > 0$ and for all $t > 0$.
- That condition is equivalent to say that **a linear time invariant system is Bounded Input Bounded State stable (BIBS stable) if and only if all the eigenvalues of the system matrix $\mathbf{A}$ have negative real part** (in that case the matrix of the system $\mathbf{A}$ is said to be Hurwitz).

Universidad Pontificia Bolivariana

Notes

BIBS stability for LTI systems

- Notice that there could be some linear time invariant systems which are Bounded Input Bounded Output stable but are Bounded Input Bounded State unstable. However all linear time invariant systems which are Bounded Input Bounded State stable are also Bounded Input Bounded State stable.
- This is due to the fact that not necessarily all the eigenvalues of the system matrix $\mathbf{A}$ are system poles, however all the system poles are eigenvalues of the system matrix $\mathbf{A}$.

Universidad Pontificia Bolivariana

Notes

Routh-Hurwitz stability criterion

- Routh-Hurwitz stability criterion establish an easy way to check whether all roots of a polynomial have negative real part or not.
- Consider the characteristic equation
  $P(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0 = 0$
- First check that all coefficients a_i for $i = 1, \ldots, n$ are present and have the same sign, **if that is not true then not all the roots have negative real part**.
- If previous condition is met, then construct the Routh array

$$
\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
s^{n-2} & b_1 & b_2 & b_3 & \ldots \\
s^{n-3} & c_1 & c_2 & c_3 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \\
s^0 & & & &
\end{array}
$$

Universidad Pontificia Bolivariana

Notes

Routh-Hurwitz stability criterion

where

$$
\begin{aligned}
b_1 =& \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \\
b_2 =& \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \\
& \vdots \\
c_1 =& \frac{b_1 a_{n-3} - a_{n-1}b_2}{b_1} \\
c_2 =& \frac{b_1 a_{n-5} - a_{n-1}b_3}{b_1} \\
& \vdots
\end{aligned}
$$

Universidad Pontificia Bolivariana

Notes

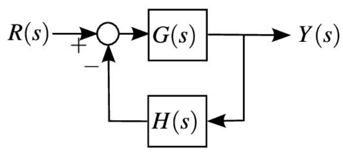
Routh-Hurwitz stability criterion

- If a zero is obtained in the first column of any row, it can be changed by a small number, ϵ, and continue the procedure normally.
- Check the signs in the first column of Routh array.
- **If there are no sign changes in the first column then all the roots have negative real part.**

Notes

Nyquist stability criterion

Consider the closed loop system shown in the figure

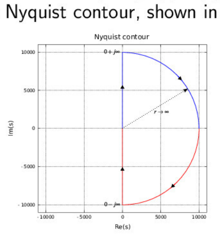


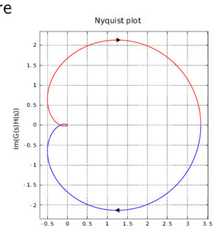
- The open loop transfer function, the transfer function of the loop when the feedback loop is broken, is $G(s)H(s)$.
- Given that $\lim_{\omega \to \infty} G(j\omega)H(j\omega) = constant$, there is a relationship among the number of zeros (Z) and poles (P) of $1 + G(s)H(s)$ in the right hand side of the complex plane and the number of clock-wise encirclements of $-1 + j0$ of the Nyquist plot (N).

Notes

Nyquist stability criterion

The Nyquist plot is the result of plotting $G(s)H(s)$ in the $G(s)H(s)$ complex plane ($Im\,(G(s)H(s))$ vs. $Re\,(G(s)H(s))$) when s follows the Nyquist contour, shown in the figure





$$N = Z - P$$

Notes

Outline

Notes

Observability: the basic idea

- A system is said to be **observable** if it is possible to know the state of the system based on the knowledge of the input and the output of the system.
- If a system is observable, that means that it is possible to estimate the state of the system based on its outputs using an **observer** or a **state estimator**.

Notes

Observability for linear time invariant systems

- A linear time invariant system can be repressented by its Jordan canonical form

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} \lambda_1 & 0 & 0 & & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & & 0 \\ & \vdots & & & \\ 0 & 0 & 0 & & \lambda_n \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & & b_{1,m} \\ b_{2,1} & b_{2,2} & b_{2,3} & \dots & b_{2,m} \\ b_{3,1} & b_{3,2} & b_{3,3} & & b_{3,m} \\ & \vdots & & & \\ b_{n,1} & b_{n,2} & b_{n,3} & & b_{n,m} \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & & c_{1,n} \\ c_{2,1} & c_{2,2} & c_{2,3} & \dots & c_{2,n} \\ c_{3,1} & c_{3,2} & c_{3,3} & & c_{3,n} \\ & \vdots & & & \\ c_{p,1} & c_{p,2} & c_{p,3} & & c_{p,n} \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & & d_{1,m} \\ d_{2,1} & d_{2,2} & d_{2,3} & \dots & d_{2,m} \\ d_{3,1} & d_{3,2} & d_{3,3} & & d_{3,m} \\ & \vdots & & & \\ d_{p,1} & d_{p,2} & d_{p,3} & & d_{p,m} \end{bmatrix} \mathbf{u}(t)$$

- For all the system modes to be **observable**, all the state variables must be viewed from the outputs, meaning that $\mathbf{C}_n$ **must not have null columns**.

Notes

Observability for linear time invariant systems

- A general check of observability is obtained by looking at the rank of the observability matrix defined as

$$\mathscr{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \dots \\ \mathbf{CA}^{n-1} \end{bmatrix} \quad (1)$$

- It can be shown that a system is observable if and only if the observability matrix is full rank, meaning that $rank\,(\mathscr{O}) = n$.

Notes

Outline

Notes

Controllability: the basic idea

- A system is said to be **controllable** if it is always possible to determine an input signal that carries the state from any initial condition to zero. For linear time invariant systems that is equivalent to say that the system is said to be **controllable** if it is always possible to determine an input signal that carries the state from any initial condition to any desired state, meaning that the system can always be controlled.
- If a system is controllable, that means that it is possible to control the state of the system through its inputs using a **controller** or a **control system**.

Notes

Controllability for linear time invariant systems

- A linear time invariant system can be represented by its Jordan canonical form

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} \lambda_1 & 0 & 0 & & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & & 0 \\ & \vdots & & & \\ 0 & 0 & 0 & & \lambda_n \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & & b_{1,m} \\ b_{2,1} & b_{2,2} & b_{2,3} & \dots & b_{2,m} \\ b_{3,1} & b_{3,2} & b_{3,3} & & b_{3,m} \\ & \vdots & & & \\ b_{n,1} & b_{n,2} & b_{n,3} & & b_{n,m} \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & & c_{1,n} \\ c_{2,1} & c_{2,2} & c_{2,3} & \dots & c_{2,n} \\ c_{3,1} & c_{3,2} & c_{3,3} & & c_{3,n} \\ & \vdots & & & \\ c_{p,1} & c_{p,2} & c_{p,3} & & c_{p,n} \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & & d_{1,m} \\ d_{2,1} & d_{2,2} & d_{2,3} & \dots & d_{2,m} \\ d_{3,1} & d_{3,2} & d_{3,3} & & d_{3,m} \\ & \vdots & & & \\ d_{p,1} & d_{p,2} & d_{p,3} & & d_{p,m} \end{bmatrix} \mathbf{u}(t)$$

- For all the system modes to be **controllable**, all the state variables must be connected to any of the inputs, meaning that $\mathbf{B}_n$ **must not have null rows**.

Notes

Controllability for linear time invariant systems

- A general check of controllability is obtained by looking at the rank of the controllability matrix defined as

$$\mathscr{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \tag{2}$$

- It can be shown that a system is controllable if and only if the controllability matrix is full rank, meaning that $rank\left(\mathscr{C}\right) = n$.

Notes

Outline

Notes

Examples

Determine if the following systems are BIBO stable or not.

1. Systems modeled by the following differential equations, where $u(t)$ is the input signal and $y(t)$ is the output signal.
   1. $\left(D^2 - 2D + 1\right) y(t) = (D + 1)\, u(t)$.
   2. $\left(D^4 + 6D^3 + 11D^2 + 6D\right) y(t) = \left(D^2 + 8D + 16\right) u(t)$.
2. systems modeled by the following transfer functions, where $U(s)$ is the Laplace transform of the input signal and $Y(s)$ is the Laplace transform of the output signal.
   1. $H(s) = \dfrac{Y(s)}{U(s)} = \dfrac{s^2 + 3s + 7}{(s+1)(s-3)(s+2)}$.
   2. $H(s) = \dfrac{Y(s)}{U(s)} = \dfrac{s\left(s^2 + 6s + 2\right)}{s^3 + s^2 + 5s + 1}$.
   3. $H(s) = \dfrac{Y(s)}{U(s)} = \dfrac{s^2 + 2s + 10}{2s^5 + s^4 + 8s^3 + 6s^2 + 2s + 12}$.

Universidad Pontificia Bolivariana

Notes

Examples

Consider the systems modeled in state space by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

where $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ is the input vector and $\mathbf{y}(t)$ is the output vector, with

1. $\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 9 & 0 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.
2. $\mathbf{A} = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 5 & 1 \\ 9 & 0 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix}$.

For each case do the following

- i. Obtain the transfer matrix of the system.
- ii. Determine if the system is BIBO stable.
- iii. Determine if the system is BIBS stable.
- iv. Determine if the system is observable.
- v. Determine if the system is controllable.

Universidad Pontificia Bolivariana

Notes

Notes

Notes