Automatic Flight Control

Frequency Response of Linear Time-Invariant Systems

Luis Benigno Gutiérrez Zea

Facultad de Ingeniería Aeronáutica
Universidad Pontificia Bolivariana

First semester - 2025

Notes

Outline

Notes

Outline

Notes

Frequency response of linear time invariant systems

- Consider a linear time invariant system with transfer function

$$G(s)=\frac{Y(s)}{U(s)}$$

where $u(t)$ is the input signal and $y(t)$ is the output signal, with $U(s)=\mathscr{L}\{u(t)\}$ and $Y(s)=\mathscr{L}\{y(t)\}$.

- Remember that $G(s)$ is calculated assuming zero initial conditions.

Notes

Frequency response of linear time invariant systems

- For the case of inputs of the form $u(t) = e^{st}$ it is known that the forced response, $y_f(t)$, can be calculated using the transfer function:

$$y_f(t) = G(s)e^{st}$$

- It is also known that the for an input of the form $u(t) = Ae^{\sigma t}Cos(\omega t + \phi)$, the forced response, $y_f(t)$, can be calculated using the transfer function as:

$$y_f(t) = A|G(\sigma + j\omega)|e^{\sigma t}Cos(\omega t + \phi + \angle G(\sigma + j\omega))$$

Notes

Frequency response of linear time invariant systems

- For the special case of pure sinusoidal inputs of the form $u(t) = ACos(\omega t + \phi)$, the forced response, $y_f(t)$, can be calculated using the transfer function as:

$$y_f(t) = A|G(j\omega)|Cos(\omega t + \phi + \angle G(j\omega))$$

- $\omega = \frac{2\pi}{T}$ is the angular frequency of the sinusoidal signal (in rad/s) where T is its period.
- For a **stable system**, if $u(t) = ACos(\omega t + \phi)$ for all t, then the total output is the same forced response or the steady state output of the system $y(t) = A|G(j\omega)|Cos(\omega t + \phi + \angle G(j\omega))$

Notes

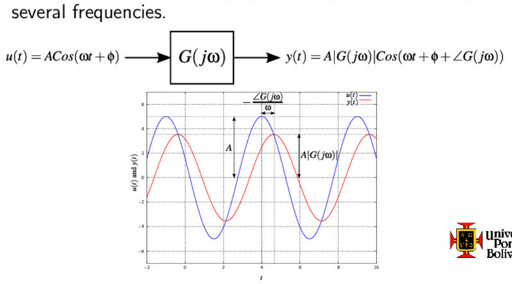
Frequency response of linear time invariant systems

- Note that when the input signal is a sinusoidal signal, the output signal is a sinusoidal signal with the same frequency.
- There are two modifications in the output signal:
  - The amplitude is multiplied by $|G(j\omega)|$.
  - The phase is shifted in a value $\angle G(j\omega)$.
- That is why $G(j\omega)$ is called the **frequency response** of the system.
- $|G(j\omega)|$ is the **amplitude response** of the system. $|G(j\omega)|$ is sometimes called the **gain** of the system.
- $\angle G(j\omega)$ is the **phase response** of the system. $\angle G(j\omega)$ is sometimes called the **phase** of the system.

Notes

Frequency response of linear time invariant systems

- The frequency response can be easily measured experimentally measuring the ratio of amplitudes and the phase shift between the input and output applying at the input sinusoidal signals of several frequencies.



Notes

Outline

Universidad Pontificia Bolivariana

Notes

Bode plot

- The **Bode plot** is a plot of the **frequency response** of a linear time invariant system that is presented in two plots in the following way:
  - The **amplitude Bode plot** is the plot of the **amplitude response** of the system as a function of the frequency, using a logarithmic scale for the amplitude and a logarithmic scale for the frequency.
  - The **phase Bode plot** is the plot of the **phase response** of the system as a function of the frequency, using a linear scale for the phase and a logarithmic scale for the frequency.

Universidad Pontificia Bolivariana

Notes

Amplitude Bode plot

- The amplitude is plotted using a logarithmic scale in **decibels**, as a function of the frequency, using a logarithmic scale for the frequency.

$$|G(j\omega)|_{dB} = 20log\left(|G(j\omega)|\right)$$

Universidad Pontificia Bolivariana

Notes

Phase Bode plot

- The phase is plotted using a linear scale for the phase angle as a function of the frequency, using a logarithmic scale for the frequency.

$$\angle G(j\omega)$$

Universidad Pontificia Bolivariana

Notes

Drawing a Bode plot

- Factoring the transfer function numerator and denominator as

$$G(s) = \frac{Z_1(s)Z_2(s)\ldots Z_p(s)}{P_1(s)P_2(s)\ldots P_q(s)}$$

- The amplitude Bode plot will be

$$\begin{aligned}|G(j\omega)|_{dB} =& 20log\left(|G(j\omega)|\right)\\ =& \sum_{k=1}^{p} 20log\left(|Z_k(j\omega)|\right) - \sum_{i=1}^{q} 20log\left(|P_i(j\omega)|\right)\\ =& \sum_{k=1}^{p} |Z_k(j\omega)|_{dB} - \sum_{i=1}^{q} |P_i(j\omega)|_{dB}\end{aligned}$$

- The phase Bode plot will be

$$\angle G(j\omega) = \sum_{k=1}^{p} \angle Z_k(j\omega) - \sum_{i=1}^{q} \angle P_i(j\omega)$$

Notes

Drawing a Bode plot

- Therefore a Bode plot can be obtained summing up the terms associated with the factors in the numerator and subtracting the terms associated with the factors in the denominator in amplitude expressed in decibels and in phase.
- It is enough to know how to draw the amplitude and phase bode plots for the following cases:
  - Case 1: a constant, K
  - Case 2: a delay, e^{-st_o}
  - Case 3: a power of s, s^k
  - Case 4: a power of terms of first order, $(\tau s + 1)^k$
  - Case 5: a power of terms of second order, $(\tau^2 s^2 + 2\zeta\tau s + 1)^k$

Notes

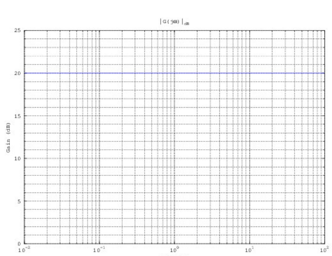
Case 1: a constant, K

$$|K|_{dB} = 20log\left(|K|\right)$$

$$\angle K = \begin{cases} 0, \ K > 0 \\ 180°, \ K < 0 \end{cases}$$

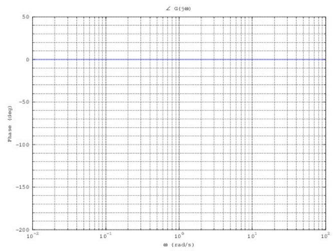
Notes

Case 1: a constant, K



Notes

Case 1: a constant, K



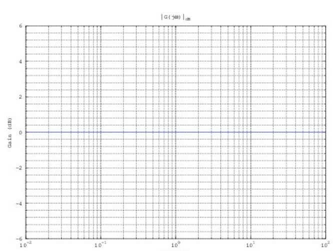
Notes

Case 2: a delay, e^{-st_o}

$$|e^{-st_o}|_{dB} = 0dB$$
$$\angle e^{-j\omega t_o} = -t_o\omega$$

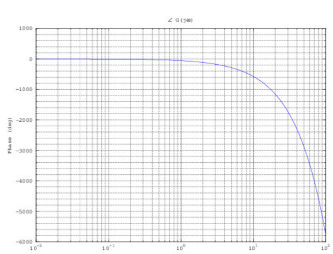
Notes

Case 2: a delay, e^{-st_o}



Notes

Case 2: a delay, e^{-st_o}



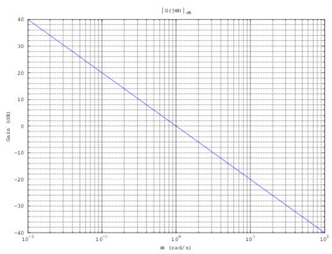
Notes

Case 3: a power of s, s^k

$$|s^k|_{dB} = 20k \cdot log\,(\omega)$$
$$\angle s^k = k90^\circ$$

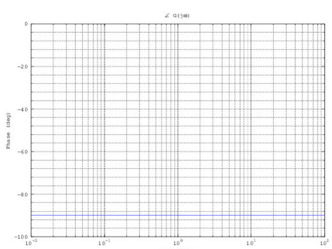
Notes

Case 3: a power of s, s^k



Notes

Case 3: a power of s, s^k



Notes

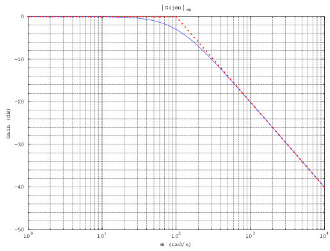
Case 4: a power of terms of first order, $(\tau s + 1)^k$

$$|\,(\tau s+1)^k\,|_{dB} = \begin{cases} 0dB,\ \omega \ll \frac{1}{\tau} \\ k \cdot 3dB,\ \omega = \frac{1}{\tau} \\ 20k \cdot log\,(\omega) - 20k \cdot log\left(\frac{1}{\tau}\right),\ \omega \gg \frac{1}{\tau} \end{cases}$$

$$\angle\,(\tau s+1)^k = \begin{cases} 0^\circ,\ \omega \ll \frac{1}{\tau} \\ k45^\circ,\ \omega = \frac{1}{\tau} \\ k90^\circ,\ \omega \gg \frac{1}{\tau} \end{cases}$$

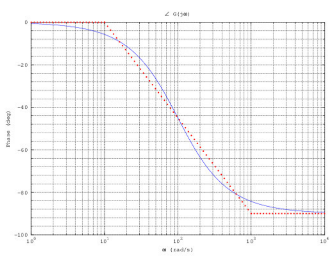
Notes

Case 4: a power of terms of first order, $(\tau s+1)^k$



Notes

Case 4: a power of terms of first order, $(\tau s+1)^k$



Notes

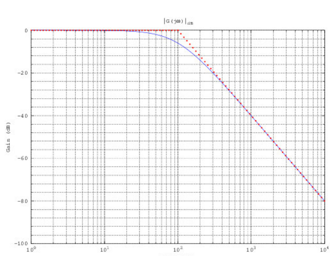
Case 5: a power of terms of second order, $\left(\tau^2s^2+2\zeta\tau s+1\right)^k$

$$\left|\left(\tau^2s^2+2\zeta\tau s+1\right)^k\right|_{dB}=\begin{cases}0dB,\ \omega\ll\frac{1}{\tau}\\ 20k\cdot log\left(2\zeta\sqrt{1-\zeta^2}\right),\ \omega=\omega_p=\frac{\sqrt{1-2\zeta^2}}{\tau}\text{ for }\zeta<\frac{\sqrt{2}}{2}\\ 20k\cdot log\left(2\zeta\right),\ \omega=\frac{1}{\tau}\\ 40k\cdot log\left(\omega\right)-40k\cdot log\left(\frac{1}{\tau}\right),\ \omega\gg\frac{1}{\tau}\end{cases}$$

$$\angle\left(\tau^2s^2+2\zeta\tau s+1\right)^k=\begin{cases}0^\circ,\ \omega\ll\frac{1}{\tau}\\ k\cdot tan^{-1}\left(\frac{2\zeta\tau\omega}{1-\omega^2\tau^2}\right),\ 0<\omega<\frac{1}{\tau}\\ k90^\circ,\ \omega=\frac{1}{\tau}\\ k\left(180^\circ-tan^{-1}\left(\frac{2\zeta\tau\omega}{\omega^2\tau^2-1}\right)\right),\ \omega>\frac{1}{\tau}\\ k180^\circ,\ \omega\gg\frac{1}{\tau}\end{cases}$$

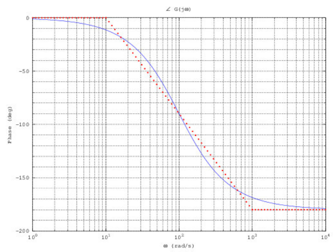
Notes

Case 5: a power of terms of second order, $\left(\tau^2s^2+2\zeta\tau s+1\right)^k$ with $\zeta=1$



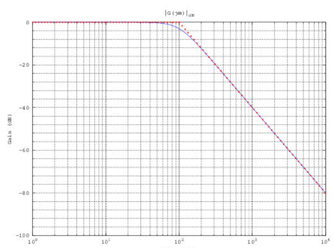
Notes

Case 5: a power of terms of second order, $\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)^k$ with $\zeta = 1$



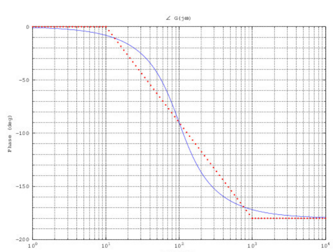
Notes

Case 5: a power of terms of second order, $\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)^k$ with $\zeta = \frac{\sqrt{2}}{2}$



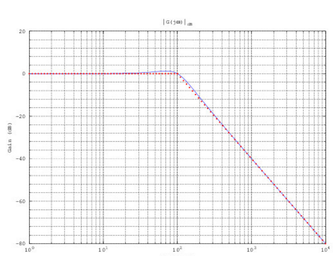
Notes

Case 5: a power of terms of second order, $\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)^k$ with $\zeta = \frac{\sqrt{2}}{2}$



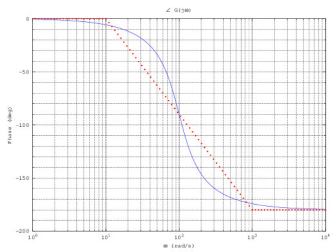
Notes

Case 5: a power of terms of second order, $\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)^k$ with $\zeta = \frac{1}{2}$



Notes

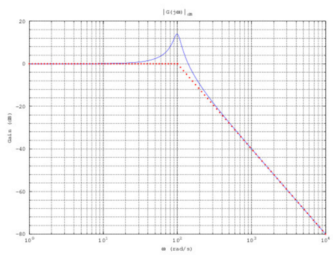
Case 5: a power of terms of second order, $\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)^k$ with $\zeta = \frac{1}{2}$



Universidad Pontificia Bolivariana

Notes

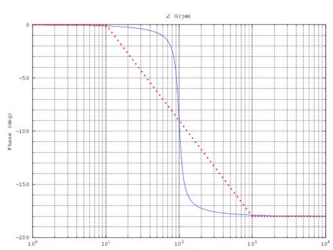
Case 5: a power of terms of second order, $\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)^k$ with $\zeta = 0.1$



Universidad Pontificia Bolivariana

Notes

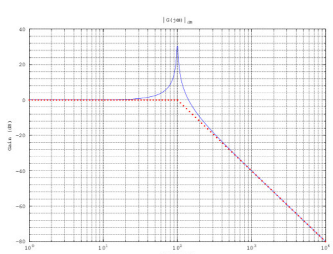
Case 5: a power of terms of second order, $\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)^k$ with $\zeta = 0.1$



Universidad Pontificia Bolivariana

Notes

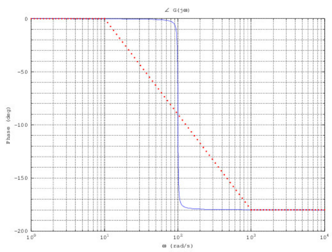
Case 5: a power of terms of second order, $\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)^k$ with $\zeta = 0.01$



Universidad Pontificia Bolivariana

Notes

Case 5: a power of terms of second order, $\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)^k$ with $\zeta = 0.01$



Notes

Outline

Notes

Other frequency response plots

- There are other plots related to the frequency response of the system that are parametric in the frequency:
  - The Nyquist plot.
  - The Nichols plot.

Notes

Nyquist plot

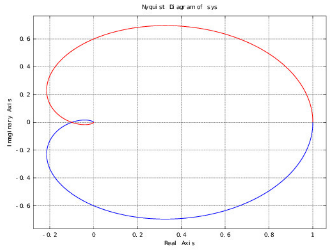
- The Nyquist plot is a plot of $G(j\omega)$ in the complex plane with the frequency as a parameter with $0 < \omega < \infty$.
- It is equivalent to a plot of $Im\left(G(j\omega)\right)$ as a function of $Re\left(G(j\omega)\right)$ in rectangular coordinates for $0 < \omega < \infty$.
- In polar coordinates is a plot of $G(j\omega)$ in the complex plane, so $|G(j\omega)|$ is the magnitude of each point and $\angle G(j\omega)$ is the angle of each point for $0 < \omega < \infty$.
- There is a important criterion for stability of feedback systems based on the Nyquist plot that will be studied later on: the Nyquist stability criterion.

Notes

Nyquist plot

- Nyquist plot for

$$G(s) = \frac{6}{(s+1)(s+2)(s+3)}$$



Notes

Nichols plot

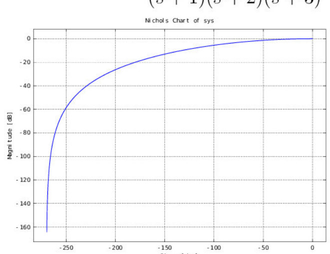
- The Nichols plot is a plot of $|G(j\omega)|_{dB}$ as a function of $\angle G(j\omega)$ for $0 < \omega < \infty$.
- There is also a way to relate the characteristics of this plot with the stability of a feedback system.

Notes

Nichols plot

- Nichols plot for

$$G(s) = \frac{6}{(s+1)(s+2)(s+3)}$$



Notes

Outline

Notes

Drawing frequency response plots in Octave/Matlab ®

- To draw the Bode plot in Octave/Matlab ® use the command *bode*.
- To draw the Nyquist plot in Octave/Matlab ® use the command *nyquist*.
- For the Nichols plot in Octave/Matlab ® use the command *nichols*.

Notes

Outline

Notes

Examples

Notes

Notes