Automatic Flight Control

Frequency Domain Analysis

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First semester - 2025

Notes

Outline I

1. Laplace transform
   - Definitions
   - Properties of the Laplace transform
   - Some useful Laplace transform pairs
2. Inverse Laplace transform
3. Examples of calculation of inverse Laplace transforms
4. Frequency domain analysis with Laplace transform
5. Examples
6. Frequency domain solution of the state space equations
7. Examples

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Laplace transform

Definitions

Direct Laplace transform

$$X(s) = \mathscr{L}\left\{x(t)\right\} = \int_{0^-}^{\infty} x(t)e^{-st}dt \qquad (1)$$

Inverse Laplace transform

$$x(t) = \mathscr{L}^{-1}\left\{X(s)\right\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds \qquad (2)$$

with $c + jw \in ROC$ for all $w \in \mathbb{R}$, where ROC is the region of convergence of the Laplace transform of $x(t)$, $X(s)$.

Notes

Properties of the Laplace transform

$x(t)$	$X(s) = \mathscr{L}\{x(t)\}$
$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$
$x(t - t_o)$	$e^{-t_o s}X(s)$
$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$
$e^{-at}x(t)$	$X(s + a)$
$tx(t)$	$-\frac{dX(s)}{ds}$
$(-t)^n x(t)$	$\frac{d^nX(s)}{ds^n}$
$\frac{x(t)}{t}$	$\int_s^\infty X(u)du$

Notes

Properties of the Laplace transform

$x(t)$	$X(s) = \mathscr{L}\{x(t)\}$
$\frac{dx(t)}{dt}$	$sX(s) - x(0^-)$
$\int_{-\infty}^{t} x(\lambda)\, d\lambda$	$\frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(t)\, dt}{s}$
$x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\, d\lambda$	$X(s)H(s)$

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Properties of the Laplace transform

Initial value theorem

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Final value theorem

$$x(\infty) = \lim_{s\to 0} sX(s)$$

Notes

Properties of the Laplace transform

Note that for the $n-th$ derivative of a signal

$$\mathscr{L}\left\{\frac{d^nx(t)}{dt^n}\right\} = s^nX(s) - s^{n-1}x(0^-) - s^{n-2}x^{(1)}(0^-) - \dots - sx^{(n-2)}(0^-) - x^{(n-1)}(0^-)$$

Notes

Some useful Laplace transform pairs

$x(t)$	$X(s) = \mathscr{L}\{x(t)\}$
$\delta(t)$	1
$1(t)$	$\frac{1}{s}$
$r(t)$	$\frac{1}{s^2}$
$p(t) = \int_{-\infty}^{t} r(\lambda)\, d\lambda = \begin{cases} \frac{t^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$	$\frac{1}{s^3}$
$t^n 1(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at} 1(t)$	$\frac{1}{s+a}$
$t^n e^{-at} 1(t)$	$\frac{n!}{(s+a)^{n+1}}$

Notes

Some useful Laplace transform pairs

$x(t)$	$X(s) = \mathscr{L}\{x(t)\}$
$Cos\,(w_o t)\,1(t)$	$\frac{s}{s^2 + w_o^2}$
$Sin\,(w_o t)\,1(t)$	$\frac{w_o}{s^2 + w_o^2}$
$e^{-at} Cos\,(w_o t)\,1(t)$	$\frac{s+a}{(s+a)^2 + w_o^2}$
$e^{-at} Sin\,(w_o t)\,1(t)$	$\frac{w_o}{(s+a)^2 + w_o^2}$
$te^{-at} Cos\,(w_o t)\,1(t)$	$\frac{(s+a)^2 - w_o^2}{\left[(s+a)^2 + w_o^2\right]^2}$
$te^{-at} Sin\,(w_o t)\,1(t)$	$\frac{2w_o\,(s+a)}{\left[(s+a)^2 + w_o^2\right]^2}$

Notes

Some useful Laplace transform pairs

- Next Laplace transform pairs are valid for $0 \leq \zeta < 1$.

$x(t)$	$X(s) = \mathscr{L}\{x(t)\}$
$\frac{w_n}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} Sin\left(w_n \sqrt{1-\zeta^2}\, t\right) 1(t)$	$\frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$
$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} Sin\left(w_n \sqrt{1-\zeta^2}\, t - Tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right) 1(t)$	$\frac{s}{s^2 + 2\zeta w_n s + w_n^2}$
$\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} Sin\left(w_n \sqrt{1-\zeta^2}\, t + Tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)\right] 1(t)$	$\frac{w_n^2}{s\,(s^2 + 2\zeta w_n s + w_n^2)}$

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Inverse Laplace transform

- To calculate the inverse Laplace transform the definition given by equation (2) can be used, with complex variable theory tools.
- Or the complex variable theory integral can be avoided using the properties of the Laplace transform, and a table of Laplace transform pairs.
- I.e. the inverse Laplace transform of

$$\begin{aligned} Y(s) =& \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p_1)(s-p_2)\ldots(s-p_n)} \\ =& \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \ldots + \frac{c_n}{s-p_n}, \end{aligned}$$

- is $y(t) = \left(c_1 e^{p_1 t} + c_2 e^{p_2 t} + \ldots + c_n e^{p_n t}\right) 1(t)$
- where $c_i = \frac{N(s)}{D(s)} (s-p_i)|_{s=p_i} \quad i = 1, \ldots, n$

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Examples of calculation of inverse Laplace transforms

Find the the inverse laplace transform in each case.

1. $X(s) = 4\left(1 - e^{-st_o}\right)$.
2. $Y(s) = \frac{K}{\tau s + 1}$.
3. $Z(s) = \frac{Ks}{s + w_o}$.
4. $Y(s) = \frac{s+1}{(s+2)(s+3)}$.
5. $Y(s) = \frac{(s^2+4s+3)\, e^{-3s}}{s^3+3s^2+3s+2}$.
6. $Y(s) = \frac{(s^2+5s+6)}{(s+5)(s^2+s+1)}$.

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Frequency domain analysis with Laplace transform

- For a linear time invariant system modeled by

$$\left(a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \ldots + a_2 D^2 + a_1 D + a_0\right) y(t) = \left(b_m D^m + b_{m-1} D^{m-1} + \ldots + b_1 D + b_0\right) u(t),$$

- The transfer function is given by

$$\frac{Y(s)}{U(s)} = \frac{\mathscr{L}\{y(t)\}}{\mathscr{L}\{u(t)\}} = \frac{b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \ldots + b_2 s^2 + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_2 s^2 + a_1 s + a_0}$$

assuming null initial conditions.

Notes

Frequency domain analysis with Laplace transform

- If we denote the transfer function of the system by $H(s)$ we get

$$H(s) = \frac{Y(s)}{U(s)} = \left.\frac{\mathscr{L}\{y(t)\}}{\mathscr{L}\{u(t)\}}\right|_{Null\ IC} \tag{3}$$

- Therefore, we can write

$$Y(s) = \frac{Y(s)}{U(s)} U(s) = H(s)U(s) \tag{4}$$

- That is, **the Laplace transform of the output signal is obtained by multiplying the transfer function of the system times the Laplace transform of the input signal.**

Notes

Frequency domain analysis with Laplace transform

- Other way to see the same.
- The linear system can be modeled by a linear operator $\mathscr{H}$ such that $y(t) = \mathscr{H}\{u(t)\}$.
- The input signal can be represented by its Laplace transform using equation (2), i.e.

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds.$$

Notes

Frequency domain analysis with Laplace transform

- Therefore, using the linearity of the system we get

$$\begin{aligned} y(t) =& \mathscr{H}\{x(t)\} \\ =& \mathscr{H}\left\{\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds\right\} \\ =& \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s) \mathscr{H}\left\{e^{st}\right\} ds \\ =& \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s) H(s) e^{st} ds \\ =& \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} Y(s) e^{st} ds \end{aligned}$$

- Meaning that $Y(s) = H(s)U(s)$, the same than before!

Notes

Frequency domain analysis with Laplace transform

- Another way to see the same.
- The output of a linear time invariant system can be calculated using the **convolution integral**

$$y(t) = \mathscr{H}\{u(t)\} = u(t) * h(t) = \int_{-\infty}^{\infty} u(\lambda)h(t-\lambda)d\lambda$$

- where $h(t) = \mathscr{H}\{\delta(t)\}$ is the impulse response of the system.
- Using convolution property of Laplace transform we get

$$Y(s) = \mathscr{L}\{y(t)\} = \mathscr{L}\{u(t) * h(t)\} = \mathscr{L}\{u(t)\}\mathscr{L}\{h(t)\}$$

- But $\mathscr{L}\{u(t)\} = U(s)$ and $\mathscr{L}\{h(t)\} = H(s)$
- Therefore $Y(s) = U(s)H(s) = H(s)U(s)$, the same than before!

Notes

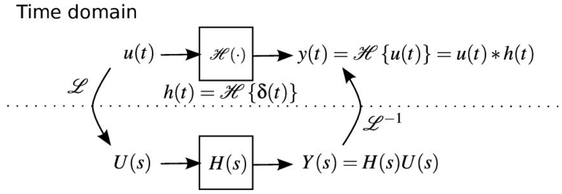
Frequency domain analysis with Laplace transform

- The use of equations (3) and (4) together with the use of algebraic methods and a Laplace transform table, makes the Laplace transform a very handy tool for the analysis of linear time invariant systems.

Notes

Frequency domain analysis with Laplace transform

Time domain



Frequency domain

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Notes

Example 1: step response of a first order system

Find the step response of a first order system modeled by

$$(\tau D + 1)\, y(t) = K u(t),$$

where $u(t)$ is the input signal and $y(t)$ is the output signal.
The step response is the response of the system when the input signal is a step, i.e. $u(t) = 1(t)$

Notes

Example 2: step response of a second order system: overdamped case

Find the step response of a second order system modeled by

$$\left(D^2 + 2\zeta\omega_n D + \omega_n^2\right) y(t) = K\omega_n^2 u(t),$$

where $u(t)$ is the input signal and $y(t)$ is the output signal. In this case assume that $\zeta > 1$ (overdamped case).

Notes

Example 2: step response of a second order system: critically damped case

Find the step response of a second order system modeled by

$$\left(D^2 + 2\zeta\omega_n D + \omega_n^2\right) y(t) = K\omega_n^2 u(t),$$

where $u(t)$ is the input signal and $y(t)$ is the output signal. In this case assume that $\zeta = 1$ (critically damped case).

Notes

Example 2: step response of a second order system: underdamped case

Find the step response of a second order system modeled by

$$\left(D^2 + 2\zeta\omega_n D + \omega_n^2\right) y(t) = K\omega_n^2 u(t),$$

where $u(t)$ is the input signal and $y(t)$ is the output signal. In this case assume that $\zeta < 1$ (underdamped case).

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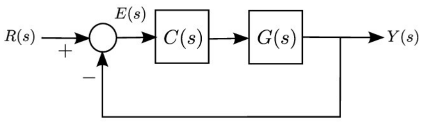
Example 3: use of initial and final value theorem

Consider the feedback control system represented by the block diagram shown in the figure.



The control system error is $e(t) = r(t) - y(t)$.

1. Determine the initial value of the error, $e\left(0^{+}\right) = \lim_{t \to 0^{+}} e(t)$, when the reference signal, $r(t)$, is a step signal.
2. Determine the steady state error, $e_{ss} = \lim_{t \to \infty} e(t)$, when the reference signal, $r(t)$, is a step signal.

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Example 4: response of a third order system

Find the response of a system modeled by

$$\left(D^{3} + 8D^{2} + 17D + 10\right) y(t) = (D + 2)\, u(t),$$

where $u(t)$ is the input signal and $y(t)$ is the output signal. Consider these cases

1. $u(t) = 1(t)$
2. $u(t) = r(t)$
3. $u(t) = e^{-t}\, 1(t)$
4. $u(t) = e^{-2t}\, 1(t)$
5. $u(t) = sin\,(5t)\; 1(t)$
6. $u(t) = e^{-t} cos\,(3t)\; 1(t)$

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Frequency domain solution of the state space equations

For linear time invariant systems the state space representation will be of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (5)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (6)$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are constant matrices.

- With $\mathbf{u}(t) \in \mathbb{R}^{m}$, $\mathbf{y}(t) \in \mathbb{R}^{p}$, $\mathbf{x}(t) \in \mathbb{R}^{n}$
- Therefore, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$.

Notes

Frequency domain solution of the state space equations

- Analyzing a system behavior in the frequency domain is finding the solution of the state equations, $\mathbf{X}(s) = \mathscr{L}\{\mathbf{x}(t)\}$ given that $\mathbf{U}(s) = \mathscr{L}\{\mathbf{u}(t)\}$ is known and the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ is known.
- For simplicity t_0 is selected as zero.
- So, the analysis problem in the frequency domain is to solve the equation (5) for $\mathbf{X}(s)$ given the input vector $\mathbf{U}(s)$ and the initial state $\mathbf{x}(0) = \mathbf{x}_0$.
- After solving for the state vector in the frequency domain, $\mathbf{X}(s)$, the output vector, $\mathbf{Y}(s)$, can be determined from output equation (6).

Notes

Frequency domain solution of the state space equations

Applying the Laplace transform to the state space equation, equation (5), we get

$$\mathscr{L}\{\dot{\mathbf{x}}(t)\} = \mathscr{L}\{\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\}$$
$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

Therefore,

$$(s\mathbf{I} - \mathbf{A})\,\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

Solving for $\mathbf{X}(s)$,

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

And replacing in the output equation, equation (6), after taking the Laplace transform, the output is obtained

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

Notes

Frequency domain solution of the state space equations

In summary

$$\mathbf{X}(s) = \mathbf{\Phi}(s)\mathbf{x}(0) + \mathbf{\Phi}(s)\mathbf{B}\mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{C}\mathbf{\Phi}(s)\mathbf{x}(0) + \mathbf{H}(s)\mathbf{U}(s)$$

where

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$$
$$\mathbf{H}(s) = \mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + \mathbf{D}$$

- $\mathbf{\Phi}(s)$ is the **state transition matrix** of the system.
- $\mathbf{H}(s)$ is the **transfer matrix** of the system.

Notes

Frequency domain solution of the state space equations

In summary

To find the time domain representation of the state and the output take the inverse Laplace transform

$$\mathbf{x}(t) = \mathscr{L}^{-1}\{\mathbf{X}(s)\} = \mathscr{L}^{-1}\{\mathbf{\Phi}(s)\mathbf{x}(0) + \mathbf{\Phi}(s)\mathbf{B}\mathbf{U}(s)\} \quad (7)$$
$$\mathbf{y}(t) = \mathscr{L}^{-1}\{\mathbf{Y}(s)\} = \mathscr{L}^{-1}\{\mathbf{C}\mathbf{\Phi}(s)\mathbf{x}(0) + \mathbf{H}(s)\mathbf{U}(s)\} \quad (8)$$

Notes

Retationship with time domain solution of the state space equations

Te state space model solution in the time domain for the state vector is given by

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t \mathbf{e}^{\mathbf{A}(t-\lambda)}\mathbf{B}\mathbf{u}(\lambda)d\lambda. \quad (9)$$

And the output of the system is obtained as

$$\mathbf{y}(t) = \mathbf{C}\mathbf{e}^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t \mathbf{C}\mathbf{e}^{\mathbf{A}(t-\lambda)}\mathbf{B}\mathbf{u}(\lambda)d\lambda + \mathbf{D}\mathbf{u}(t). \quad (10)$$

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Retationship with time domain solution of the state space equations

Note that applying the inverse Laplace transform to equations (7) and (8), equations (9) and (10) are obtained taking into account that

$$\mathscr{L}\left\{\mathbf{e}^{\mathbf{A}t}\right\} = (s\mathbf{I}-\mathbf{A})^{-1} \quad (11)$$

and

$$\mathscr{L}^{-1}\left\{(s\mathbf{I}-\mathbf{A})^{-1}\right\} = \mathbf{e}^{\mathbf{A}t} \quad (12)$$

Therefore the state transition matrix in the frequency domain is

$$\mathbf{\Phi}(s) = (s\mathbf{I}-\mathbf{A})^{-1}, \quad (13)$$

and the same state transition matrix in the time domain is

$$\phi(t) = \mathbf{e}^{\mathbf{A}t}, \quad (14)$$

such that $\mathbf{\Phi}(s) = \mathscr{L}\left\{\phi(t)\right\}$ and $\phi(t) = \mathscr{L}^{-1}\left\{\mathbf{\Phi}(s)\right\}$.

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Examples I

Consider the linear time invariant system modeled by the differential equation

$$\left(D^2+5D+6\right)y(t) = (D+1)\,u(t)$$

For this system do the following

1. Obtain the observable canonical form of the state space model.
2. Obtain the controllable canonical form of the state space model.
3. Obtain the Jordan canonical form starting from the observable canonical form of the state space model.
4. Obtain the Jordan canonical form starting from the controllable canonical form of the state space model.
5. For each form of the state space model do the followi

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Examples II

1. Calculate the state transition matrix of the system in the frequency domain.
2. Calculate the transfer transfer matrix of the system.
3. Using frequency domain methods calculate the state and output responses of the system when the input is a step signal, $u(t) = 1(t)$, and the initial state is zero, $\mathbf{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. Do this using each of the three forms obtained for the state space model of the system.
4. Using frequency domain methods calculate the state and output responses of the system when the input is zero, $u(t) = 0$, and the initial state is, $\mathbf{x}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. Do this using each of the three forms obtained for the state space model of the system.

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