Automatic Flight Control

Time Domain Analysis

Luis Benigno Gutiérrez Zea

Facultad de Ingeniería Aeronáutica
Universidad Pontificia Bolivariana

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Notes

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1. Mathematical model of linear time invariant systems
2. Analysis of linear time invariant models in time domain
3. Solution of ordinary differential equations
4. Analytical solution of ordinary differential equations
   - Meaning of natural and forced responses
   - Natural response
   - Forced response
     - Undetermined coefficients/annihilator method
     - Transfer function method
5. Examples
6. Time domain solution of the state space equations
7. Examples

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Mathematical model of linear time invariant systems

We have seen that the mathematical model for a single input single output, dynamical, lumped parameter, linear time invariant system can be:

- A differential equation (of any order).
- A state space model (a set of first order differential equations).
- A transfer function representing the differential equation (relationship between the Laplace transforms of the output and the input of the system).

Notes

Mathematical model of linear time invariant systems

Using the notation $D \triangleq \dfrac{d}{dt}$ and $\int \triangleq \int_{-\infty}^{t} (\cdot) d\lambda$, the mathematical model of a dynamical system, with input signal $u(t)$ and output signal $y(t)$, is represented by an ordinary differential equation of the form

$$\left(a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \ldots + a_2 D^2 + a_1 D + a_0\right) y(t) = \left(b_m D^m + b_{m-1} D^{m-1} + \ldots + b_1 D + b_0\right) u(t),$$

where $m \leq n$ for causal systems.

Notes

Mathematical model of linear time invariant systems

Using the operator notation

$$L_y\left(y(t)\right) = \left(a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \ldots + a_2 D^2 + a_1 D + a_0\right) y(t)$$

and

$$L_u\left(u(t)\right) = \left(b_m D^m + b_{m-1} D^{m-1} + b_{m-2} D^{m-2} + \ldots + b_2 D^2 + b_1 D + b_0\right) u(t),$$

we can write the differential equation as

$$L_y\left(y(t)\right) = L_u\left(u(t)\right) \quad (1)$$

Notes

Mathematical model of linear time invariant systems

- The model represented by equation (1) is linear since operators L_y and L_u are linear.
- That is: $L_y\left(c_1 y_1(t) + c_2 y_2(t)\right) = c_1 L_y\left(y_1(t)\right) + c_2 L_y\left(y_2(t)\right)$ for all c_1, c_2, $y_1(t)$, and $y_2(t)$; and $L_u\left(c_1 u_1(t) + c_2 u_2(t)\right) = c_1 L_u\left(u_1(t)\right) + c_2 L_u\left(u_2(t)\right)$ for all c_1, c_2, $u_1(t)$, and $u_2(t)$
- The model represented by equation (1) is invariant since operators L_y and L_u are invariant (respect to time t).
- That is: if $L_y\left(y(t)\right) = \alpha(t)$, then $L_y\left(y(t-t_0)\right) = \alpha(t-t_0)$ for all $y(t)$ and t_0; and if $L_u\left(u(t)\right) = \beta(t)$, then $L_u\left(u(t-t_0)\right) = \beta(t-t_0)$ for all $u(t)$ and t_0

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Analysis of linear time invariant models in time domain

Problem
Given the mathematical model of a linear time invariant system represented by equation (1), and given the input signal $u(t)$ for all t, determine the output signal $y(t)$ for all t (the solution of the differential equation given by equation (1)).

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Solution of ordinary differential equations

The solution of equation (1) can be obtained:

- Analytically.
- Numerically.

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Analytical solution of ordinary differential equations

The solution of equation (1) is split in two terms:

$$y(t) = y_n(t) + y_f(t) \tag{2}$$

where $y_n(t)$ is the **natural response** of the system (the homogeneous solution), which is the solution of

$$L_y\left(y_n(t)\right) = 0 \tag{3}$$

and $y_f(t)$ is the **forced response** of the system (the particular solution), which is a particular solution of

$$L_y\left(y_f(t)\right) = L_u\left(u(t)\right) \tag{4}$$

Notice that adding equations (3) and (4) we obtain equation (1), (using the linearity of L_y and L_u and equation (2))

Notes

Meaning of natural and forced responses

- The **natural response** is the way in which the system responds due to its internal dynamics and represents the way internal energy is exchanged within system components.
- The **forced response** is the way in which its internal dynamics is forced to respond due to the input signal.

Notes

Natural response

- To solve for the **natural response** we need to solve homogeneous equation (3), which states the a linear combination of $y_n(t)$ and its derivatives is zero, meaning that $y_n(t)$ and its first n derivatives should be linearly dependent functions.
- Candidate functions that fulfill that conditions are exponential functions (of the form $e^{\sigma t}$), sinusoidal functions (of the form $cos(\omega t)$ or $sin(\omega t)$), and exponential times sinusoidal functions (of the form $e^{\sigma t}cos(\omega t)$ or $e^{\sigma t}sin(\omega t)$).
- Using Euler's identity ($e^{j\theta} = cos(\theta) + jsin(\theta)$) it is possible to show that all these functions can be writen as combinations of functions of the exponential form e^{rt}, allowing r to be a complex number ($r \in \mathbb{C}$)
- Using these facts we assume that that $y_n(t) = e^{rt}$ for some $r \in \mathbb{C}$.

Notes

Natural response

- Replacing $y_n(t) = e^{rt}$ in equation (3) we get
$$L_y\left(y_n(t)\right) = L_y\left(e^{rt}\right) = 0$$
- But noticing that $D^k e^{rt} = r^k e^{rt}$, we get
$$L_y\left(y_n(t)\right) = \left(a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0\right) e^{rt} = 0$$
- Implying that
$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0. \tag{5}$$
- Equation (5) is the **characteristic equation of the system**. The solutions of this equation, $p_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$, are the **system roots** or **system poles**.

Notes

Natural response

- Notice that for a system modeled by equation (1), the transfer function is given by
$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \ldots + b_2 s^2 + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_2 s^2 + a_1 s + a_0}$$
- Denoting the numerator of the transfer function as $N(s)$ and the denominator as $D(s)$, such that

$$N(s) = b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \ldots + b_2 s^2 + b_1 s + b_0$$
$$D(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_2 s^2 + a_1 s + a_0$$

Notes

Natural response

- The **characteristic polynomial of the system** can be defined as
$P(s) = D(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_2 s^2 + a_1 s + a_0.$
- Therefore, the **characteristic equation** given by (5) can also be obtained from the transfer function as

$$P(s) = D(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_2 s^2 + a_1 s + a_0 = 0. \quad (6)$$

Notes

Natural response

- The **system poles** are the roots of the **characteristic polynomial of the system** or the solutions of the **characteristic equation of the system**
- Note that if p_i is a system pole then $\lim_{s \to p_i} \frac{Y(s)}{U(s)} = \infty$

Notes

Natural response

- Given that there are n poles for a system of order n, there are exactly n independent solutions of homogeneous equation (3)
- Thus, given the linearity of L_y, the general form of the natural response for the system modeled by equation (1) is

$$y_n(t) = \sum_{i=1}^{n} c_i e^{p_i t} = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \ldots + c_n e^{p_n t} \quad (7)$$

Where $p_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$, are the **system poles**, and $c_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$ are constants depending on the **initial conditions** of the system.

Notes

Natural response

- What to do for poles with multiplicity greater than one?
- When a pole p_i has multiplicity $q > 1$, then the solutions found in $y_n(t)$ will be of the form $e^{p_i t}, te^{p_i t}, t^2 e^{p_i t}, \ldots, t^{q-1} e^{p_i t}$

Notes

Natural response

- To avoid using complex values for c_i when p_i is a complex number, we can use the fact that complex roots appear in complex conjugate pairs given that the coefficients of the characteristic polynomial are real numbers.
- If $p_i = \sigma_i + j\omega_i$ and $p_{i+1} = p_i^* = \sigma_i - j\omega_i$ are a pair of complex poles we can write

$$\begin{aligned} c_i e^{p_i t} + c_{i+1} e^{p_{i+1} t} =& c_i e^{(\sigma_i + j\omega_i)t} + c_{i+1} e^{(\sigma_i - j\omega_i)t} \\ =& \left|c_i\right| e^{j\angle c_i} e^{\sigma_i t} e^{j\omega_i t} + \left|c_{i+1}\right| e^{j\angle c_{i+1}} e^{\sigma_i t} e^{-j\omega_i t} \end{aligned}$$

Notes

Natural response

- But, given that $c_{i+1} = c_i^*$, so $\left|c_{i+1}\right| = \left|c_i\right|$ and $\angle c_{i+1} = -\angle c_i$, it can be written

$$\begin{aligned} c_i e^{p_i t} + c_{i+1} e^{p_{i+1} t} =& \left|c_i\right| e^{j\angle c_i} e^{\sigma_i t} e^{j\omega_i t} + \left|c_{i+1}\right| e^{j\angle c_{i+1}} e^{\sigma_i t} e^{-j\omega_i t} \\ =& \left|c_i\right| e^{j\angle c_i} e^{\sigma_i t} e^{j\omega_i t} + \left|c_i\right| e^{-j\angle c_i} e^{\sigma_i t} e^{-j\omega_i t} \\ =& \left|c_i\right| e^{\sigma_i t} e^{j(\omega_i t + \angle c_i)} + \left|c_i\right| e^{\sigma_i t} e^{-j(\omega_i t + \angle c_i)} \\ =& \left|c_i\right| e^{\sigma_i t} \left(e^{j(\omega_i t + \angle c_i)} + e^{-j(\omega_i t + \angle c_i)}\right) \\ =& 2\left|c_i\right| e^{\sigma_i t} cos\left(\omega_i t + \angle c_i\right) \\ =& 2\left|c_i\right| cos\left(\angle c_i\right) e^{\sigma_i t} cos\left(\omega_i t\right) - 2\left|c_i\right| sin\left(\angle c_i\right) e^{\sigma_i t} sin\left(\omega_i t\right) \\ =& d_i e^{\sigma_i t} cos\left(\omega_i t\right) + d_{i+1} e^{\sigma_i t} sin\left(\omega_i t\right) \end{aligned}$$

where $d_i = 2\left|c_i\right| cos\left(\angle c_i\right)$ and $d_{i+1} = -2\left|c_i\right| sin\left(\angle c_i\right)$

Notes

Natural response

In summary

- If p_i is a real pole, write in $y_n(t)$ the term $c_i e^{p_i t}$
- If $p_i = \sigma_i + j\omega_i$ and $p_{i+1} = p_i^* = \sigma_i - j\omega_i$ are a pair of complex poles, write in $y_n(t)$ the terms $c_i e^{\sigma_i t} cos\left(\omega_i t\right) + c_{i+1} e^{\sigma_i t} sin\left(\omega_i t\right)$
- If p_i is a real pole with multiplicity $q > 1$, write in $y_n(t)$ the terms $c_i e^{p_i t}, c_{i+1} t e^{p_i t}, c_{i+2} t^2 e^{p_i t}, \ldots, c_{i+q-1} t^{q-1} e^{p_i t}$
- If $p_i = \sigma_i + j\omega_i$ and $p_{i+1} = p_i^* = \sigma_i - j\omega_i$ are a pair of complex poles with multiplicity $q > 1$, write in $y_n(t)$ the terms $c_i e^{\sigma_i t} cos\left(\omega_i t\right)$, $c_{i+1} e^{\sigma_i t} sin\left(\omega_i t\right)$, $c_{i+2} t e^{\sigma_i t} cos\left(\omega_i t\right)$, $c_{i+3} t e^{\sigma_i t} sin\left(\omega_i t\right)$, …, $c_{i+2q-2} t^{q-1} e^{\sigma_i t} cos\left(\omega_i t\right)$, $c_{i+2q-1} t^{q-1} e^{\sigma_i t} sin\left(\omega_i t\right)$

Notes

Forced response

- To solve for the **forced response** it is required to find a particular solution of equation (1), so $y_f(t)$ fulfills equation (4), repeated here

$$L_y\left(y_f(t)\right) = L_u\left(u(t)\right)$$

There are several methods to find $y_f(t)$:

- Undetermined coefficients/annihilator method.
- Variation of parameters method.
- Transfer function method for input signals of the form e^{st} (with $s \in \mathbb{C}$) or $e^{\sigma t} cos\left(\omega t + \phi\right)$.

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Notes

Undetermined coefficients/annihilator method

- Let L be an **annihilator operator** of the input signal $u(t)$, that is a differential operator, similar to L_y or L_u, such that

$$L\left(u(t)\right) = 0 \tag{8}$$

- I we apply L to the differential equation (equation (1)), we get

$$L\left(L_y\left(y_f(t)\right)\right) = L\left(L_u\left(u(t)\right)\right)$$

But

$$L\left(L_u\left(u(t)\right)\right) = L_u\left(L\left(u(t)\right)\right) = L_u\left(0\right) = 0$$

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Notes

Undetermined coefficients/annihilator method

Therefore

$$L\left(L_y\left(y_f(t)\right)\right) = 0 \tag{9}$$

- Equation (9) is a homogeneous equation that can be solved the same way that we solved for the natural response, $y_n(t)$, using equation (3).
- In the the forced response, $y_f(t)$, any components already present in $y_n(t)$ should be discarded.
- Coefficients in $y_f(t)$ are the **undetermined coefficients**, they are calculated replacing $y_f(t)$ in (4) and guaranteeing that the equation is fulfilled.

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Notes

Transfer function method for input signals of the form e^{st} (with $s \in \mathbb{C}$) or $e^{\sigma t} cos\left(\omega t + \phi\right)$

- When the input signal is of the form $u(t) = e^{st}$ with $s \in \mathbb{C}$, we can expect the forced response to be of the same form

$$y_f(t) = Ce^{st}$$

where C is an appropriate constant.

- Replacing $u(t) = e^{st}$ and $y_f(t) = Ce^{st}$ in equation (4), we obtain

$$\left(a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_2 s^2 + a_1 s + a_0\right) Ce^{st} = \left(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0\right) e^{st}$$

Therefore, given that this equation is valid for all $t \in \mathbb{R}$, it results that

$$C = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_2 s^2 + a_1 s + a_0}$$

Notes

Transfer function method for input signals of the form e^{st} (with $s \in \mathbb{C}$) or $e^{\sigma t} cos\left(\omega t + \phi\right)$

If $H(s) = \dfrac{Y(s)}{U(s)}$ is the transfer function of the system, the forced response of the system, when the input signal is of the form $u(t) = e^{st}$, will be

$$y_f(t) = H(s)e^{st}$$

Notes

Transfer function method for input signals of the form e^{st} (with $s \in \mathbb{C}$) or $e^{\sigma t} cos\left(\omega t + \phi\right)$

- If the input signal is of the form $u(t) = Ae^{\sigma t} cos\left(\omega t + \phi\right)$, note that

$$\begin{aligned} u(t) =& Ae^{\sigma t} cos\left(\omega t + \phi\right) \\ =& \frac{A}{2}e^{\sigma t}\left(e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right) \\ =& \frac{Ae^{j\phi}}{2}e^{(\sigma+j\omega)t} + \frac{Ae^{-j\phi}}{2}e^{(\sigma-j\omega)t} \end{aligned}$$

Notes

Transfer function method for input signals of the form e^{st} (with $s \in \mathbb{C}$) or $e^{\sigma t} cos\left(\omega t + \phi\right)$

- Using the linearity of the system we obtain

$$\begin{aligned} y_f(t) =& \frac{Ae^{j\phi}}{2}H\left(\sigma + j\omega\right)e^{(\sigma+j\omega)t} + \frac{Ae^{-j\phi}}{2}H\left(\sigma - j\omega\right)e^{(\sigma-j\omega)t} \\ =& \frac{Ae^{j\phi}}{2}\left|H\left(\sigma + j\omega\right)\right|e^{j\angle H(\sigma+j\omega)}e^{(\sigma+j\omega)t} + \frac{Ae^{-j\phi}}{2}\left|H\left(\sigma - j\omega\right)\right|e^{j\angle H(\sigma-j\omega)}e^{(\sigma-j\omega)t} \\ =& \frac{A\left|H\left(\sigma + j\omega\right)\right|e^{j(\phi+\angle H(\sigma+j\omega))}}{2}e^{(\sigma+j\omega)t} + \frac{A\left|H\left(\sigma + j\omega\right)\right|e^{-j(\phi+\angle H(\sigma+j\omega))}}{2}e^{(\sigma-j\omega)t} \\ =& A\left|H(\sigma + j\omega)\right|e^{\sigma t} cos\left(\omega t + \phi + \angle H(\sigma + j\omega)\right) \end{aligned}$$

- The fact that $H\left(\sigma - j\omega\right) = H\left(\sigma + j\omega\right)^{*}$ has been used.

Notes

Transfer function method for input signals of the form e^{st} (with $s \in \mathbb{C}$) or $e^{\sigma t} cos\left(\omega t + \phi\right)$

If $H(s) = \dfrac{Y(s)}{U(s)}$ is the transfer function of the system, the forced response of the system, when the input signal is of the form $u(t) = Ae^{\sigma t} cos\left(\omega t + \phi\right)$, will be

$$y_f(t) = A\left|H(\sigma + j\omega)\right|e^{\sigma t} cos\left(\omega t + \phi + \angle H(\sigma + j\omega)\right)$$

Notes

Transfer function method for input signals of the form e^{st} (with $s \in \mathbb{C}$) or $e^{\sigma t} cos(\omega t + \phi)$

Remarks

- The transfer function method to calculate the **forced response of the system** is only valid for input signals that can be decomposed in terms of the form e^{st}, $cos(\omega t + \phi)$, or $e^{\sigma t} cos(\omega t + \phi)$.
- This method cannot be used when $H(s)$ is undetermined in the specific value of s for the input signal, i.e. when this value is a system pole meaning that the input signal is of the form of one of the system natural response components (when there is **resonance**).

Notes

Transfer function method for input signals of the form e^{st} (with $s \in \mathbb{C}$) or $e^{\sigma t} cos(\omega t + \phi)$

Remarks

- Later on, the frequency domain method of analysis will generalize the use of the transfer function to calculate the whole system response (including natural and forced response) in the general case.
- When $H(s) = 0$ for the input signal, what happens is that the forced response is zero. In that case the value of s will be one of the **system zeros**, that is, a value of s that makes $H(s) = 0$.
- The **system zeros** are the roots of the numerator polynomial of $H(s)$, $N(s)$.

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Example 1: step response of a first order system

Find the step response of a first order system modeled by

$$(\tau D + 1)\, y(t) = K u(t),$$

where $u(t)$ is the input signal and $y(t)$ is the output signal.
The step response is the response of the system when the input signal is a step, i.e. $u(t) = 1(t)$

Notes

Example 2: step response of a second order system: overdamped case

Find the step response of a second order system modeled by

$$\left(D^2 + 2\zeta\omega_n D + \omega_n^2\right) y(t) = K\omega_n^2 u(t),$$

where $u(t)$ is the input signal and $y(t)$ is the output signal. In this case assume that $\zeta > 1$ (overdamped case).

Notes

Example 2: step response of a second order system: critically damped case

Find the step response of a second order system modeled by

$$\left(D^2 + 2\zeta\omega_n D + \omega_n^2\right) y(t) = K\omega_n^2 u(t),$$

where $u(t)$ is the input signal and $y(t)$ is the output signal. In this case assume that $\zeta = 1$ (critically damped case).

Notes

Example 2: step response of a second order system: underdamped case

Find the step response of a second order system modeled by

$$\left(D^2 + 2\zeta\omega_n D + \omega_n^2\right) y(t) = K\omega_n^2 u(t),$$

where $u(t)$ is the input signal and $y(t)$ is the output signal. In this case assume that $\zeta < 1$ (underdamped case).

Notes

Example 3: response of a third order system

Find the response of a system modeled by

$$\left(D^3 + 8D^2 + 17D + 10\right) y(t) = (D + 2)\, u(t),$$

where $u(t)$ is the input signal and $y(t)$ is the output signal. Consider these cases

1. $u(t) = 1(t)$
2. $u(t) = r(t)$
3. $u(t) = e^{-t}\, 1(t)$
4. $u(t) = e^{-2t}\, 1(t)$
5. $u(t) = sin\,(5t)\; 1(t)$
6. $u(t) = e^{-t} cos\,(3t)\; 1(t)$

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Time domain solution of the state space equations

For linear time invariant systems the state space representation will be of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{10}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{11}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are constant matrices.

- With $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$, $\mathbf{x}(t) \in \mathbb{R}^n$
- Therefore, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$.

Notes

Time domain solution of the state space equations

- To analyze a system behavior is to find the solution of the state equations, $\mathbf{x}(t)$ for $t \geq t_0$ given that $\mathbf{u}(t)$ is known for $t \geq t_0$ and the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ is known.
- For simplicity t_0 is selected as zero.
- So, the analysis problem is to solve the equation (10) for $\mathbf{x}(t)$ for $t \geq 0$ given the input vector $\mathbf{u}(t)$ for $t \geq 0$ and the initial state $\mathbf{x}(0) = \mathbf{x}_0$.
- After solving for the state vector, $\mathbf{x}(t)$, the output vector, $\mathbf{y}(t)$, can be determined from output equation (11).

Notes

Time domain solution of the state space equations

To solve equation (10) in time domain we split the solution for the state vector in two parts

$$\mathbf{x}(t) = \mathbf{x}_n(t) + \mathbf{x}_f(t) \tag{12}$$

Where $\mathbf{x}_n(t)$ is the **natural response** of the state, which is the solution of the homogeneous equation

$$\dot{\mathbf{x}}_n(t) = \mathbf{A}\mathbf{x}_n(t), \text{ with } \mathbf{x}_n(0) = \mathbf{x}(0) = \mathbf{x}_0 \tag{13}$$

and $\mathbf{x}_f(t)$ is the **forced response** of the state, which is the solution of the state equation with zero initial state

$$\dot{\mathbf{x}}_f(t) = \mathbf{A}\mathbf{x}_f(t) + \mathbf{B}\mathbf{u}(t), \text{ with } \mathbf{x}_f(0) = \mathbf{0} \tag{14}$$

Notes

Time domain solution of the state space equations

To solve equation (13) for the **natural response** of the state, let's consider first the following exponential matrix function

$$\phi(t) = \mathbf{e}^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \ldots \quad (15)$$

Differentiating respect to time the following is obtained

$$\begin{aligned}\dot{\phi}(t) =& \frac{d}{dt}\sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} \\ =& \sum_{k=1}^{\infty} \frac{\mathbf{A}^k k t^{k-1}}{k!} \\ =& \sum_{k=1}^{\infty} \frac{\mathbf{A}\mathbf{A}^{k-1} t^{k-1}}{(k-1)!} \\ =& \mathbf{A}\sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} \\ =& \mathbf{A}\mathbf{e}^{\mathbf{A}t}\end{aligned}$$

Notes

Time domain solution of the state space equations

With this, the solution of the homogeneous equation (13) is

$$\mathbf{x}_n(t) = \mathbf{e}^{\mathbf{A}t}\mathbf{x}_0 \quad (16)$$

since, differentiating respect to time, the following is obtained

$$\dot{\mathbf{x}}_n(t) = \mathbf{A}\mathbf{e}^{\mathbf{A}t}\mathbf{x}_0 = \mathbf{A}\mathbf{x}_n(t)$$

with $\mathbf{x}_n(0) = \mathbf{e}^{\mathbf{A}\cdot 0}\mathbf{x}_0 = \mathbf{x}_0$

Notes

Time domain solution of the state space equations

To solve equation (14) for the **forced response** of the state, let's use the parameter variation method, in which the solution is assumed to be of the form

$$\mathbf{x}_f(t) = \mathbf{e}^{\mathbf{A}t}\gamma(t) \quad (17)$$

now, differentiating respect to time, the following is obtained

$$\dot{\mathbf{x}}_f(t) = \mathbf{A}\mathbf{e}^{\mathbf{A}t}\gamma(t) + \mathbf{e}^{\mathbf{A}t}\dot{\gamma}(t),$$

and replacing in equation (14)

$$\mathbf{A}\mathbf{e}^{\mathbf{A}t}\gamma(t) + \mathbf{e}^{\mathbf{A}t}\dot{\gamma}(t) = \mathbf{A}\mathbf{e}^{\mathbf{A}t}\gamma(t) + \mathbf{B}\mathbf{u}(t)$$

Notes

Time domain solution of the state space equations

Resulting

$$\dot{\gamma}(t) = \mathbf{e}^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t).$$

Integrating respect to time, the solution for $\gamma(t)$ is obtained

$$\gamma(t) = \int_0^t \mathbf{e}^{-\mathbf{A}\lambda}\mathbf{B}\mathbf{u}(\lambda)d\lambda.$$

Replacing in equation (17), the forced response is obtained as

$$\begin{aligned}\mathbf{x}_f(t) =& \mathbf{e}^{\mathbf{A}t}\int_0^t \mathbf{e}^{-\mathbf{A}\lambda}\mathbf{B}\mathbf{u}(\lambda)d\lambda \\ =& \int_0^t \mathbf{e}^{\mathbf{A}(t-\lambda)}\mathbf{B}\mathbf{u}(\lambda)d\lambda \quad (18)\end{aligned}$$

Notes

Time domain solution of the state space equations

Replacing equations (16) and (18) in equation (12) the final solution for the state is obtained as

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t \mathbf{e}^{\mathbf{A}(t-\lambda)}\mathbf{B}\mathbf{u}(\lambda)d\lambda. \quad (19)$$

Replacing in the output equation (11) the result for the output of the system is obtained as

$$\mathbf{y}(t) = \mathbf{C}\mathbf{e}^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t \mathbf{C}\mathbf{e}^{\mathbf{A}(t-\lambda)}\mathbf{B}\mathbf{u}(\lambda)d\lambda + \mathbf{D}\mathbf{u}(t). \quad (20)$$

Notes

Time domain solution of the state space equations

In summary

$$\mathbf{x}(t) = \phi(t)\mathbf{x}_0 + \int_0^t \phi(t-\lambda)\mathbf{B}\mathbf{u}(\lambda)d\lambda,$$
$$\mathbf{y}(t) = \mathbf{C}\phi(t)\mathbf{x}_0 + \int_0^t \mathbf{h}(t-\lambda)\mathbf{u}(\lambda)d\lambda.$$

where

$$\phi(t) = \mathbf{e}^{\mathbf{A}t}$$
$$\mathbf{h}(t) = \mathbf{C}\phi(t)\mathbf{B} + \mathbf{D}\delta(t)$$

- $\phi(t)$ is the time domain representation of the **state transition matrix** of the system.
- $\mathbf{h}(t)$ is the **impulse response matrix** of the system.

Notes

Outline

1. Mathematical model of linear time invariant systems
2. Analysis of linear time invariant models in time domain
3. Solution of ordinary differential equations
4. Analytical solution of ordinary differential equations
   - Meaning of natural and forced responses
   - Natural response
   - Forced response
     - Undetermined coefficients/annihilator method
     - Transfer function method
5. Examples
6. Time domain solution of the state space equations
7. Examples

Notes

Examples I

Consider the linear time invariant system modeled by the differential equation

$$\left(D^2 + 5D + 6\right) y(t) = (D+1)\, u(t)$$

For this system do the following

1. Obtain the observable canonical form of the state space model.
2. Obtain the controllable canonical form of the state space model.
3. Obtain the Jordan canonical form starting from the observable canonical form of the state space model.
4. Obtain the Jordan canonical form starting from the controllable canonical form of the state space model.
5. For each form of the state space model do the followi

Notes

Examples II

1. Calculate the state transition matrix of the system in the time domain.
2. Calculate the transfer impulse response matrix of the system.
3. Calculate the state and output responses of the system when the input is a step signal, $u(t) = 1(t)$, and the initial state is zero, $\mathbf{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. Do this using each of the three forms obtained for the state space model of the system.
4. Calculate the state and output responses of the system when the input is zero, $u(t) = 0$, and the initial state is, $\mathbf{x}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. Do this using each of the three forms obtained for the state space model of the system.

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