Automatic Flight Control

Block Diagram Models

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Notes

Outline

1. Block diagram models
2. Block diagram models for linear time invariant systems
   - Adders
   - Series connection
   - Parallel connection
   - Feedback connection
   - Block diagrams algebra
3. Mason's gain formula
4. Examples
5. Some forms of the state space representation for linear time invariant systems
   - Observable canonical form
   - Controllable canonical form
   - Jordan canonical form

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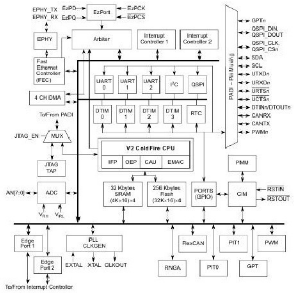
Block diagram models

- A block diagram is a graphical representation of the relationships among the parts of a system.
- Usually block diagrams are composed of blocks and directed arrows.
- Blocks represent the parts of a system (subsystems or system components).
- Directed arrows connect the blocks and represent the flow of information between blocks.
- Directed arrows can be associated with signals and blocks can be associated with systems or subsystems.
- Block diagrams can be used as a model of the system when the behavior of the blocks and the meaning of the directed arrows are clearly stated.

Notes

Block diagram models

An example of a block diagram for MCF52235 microcontroller



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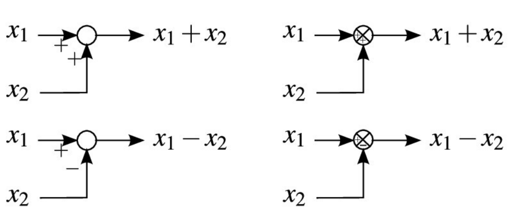
Block diagram models for linear time invariant systems

- For linear time invariant systems each block can be modeled by a set of differential equations, a state space model or a transfer function (or matrix).
- Directed arrows represent signals connecting blocks.
- Arrows going into a block represent input signals for that block.
- Arrows going out of a block represent output signals for that block.
- Arrows can be derived when the same signal goes to several blocks.

Notes

Adders



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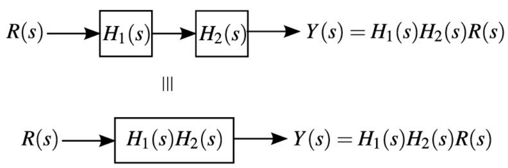
Series connection



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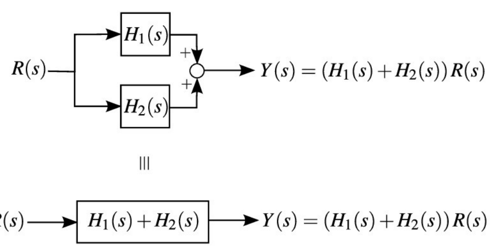
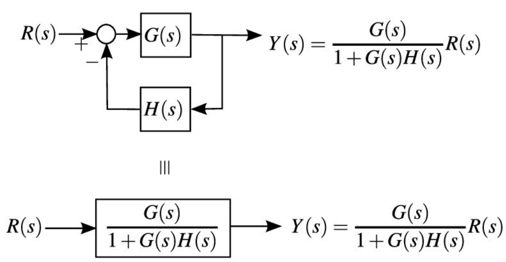
Parallel connection



Notes

Feedback connection



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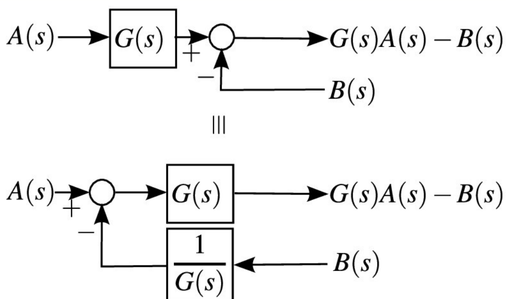
Block diagrams algebra



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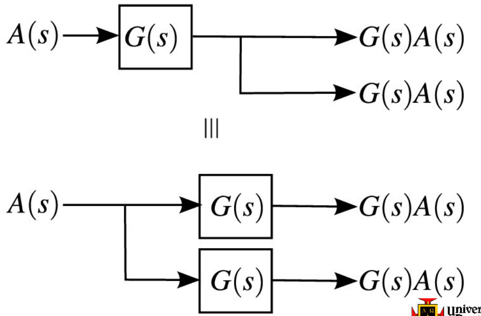
Block diagrams algebra



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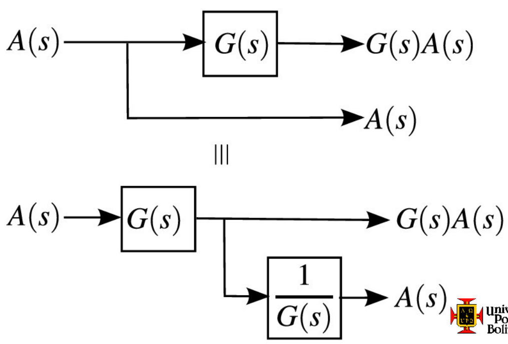
Block diagrams algebra



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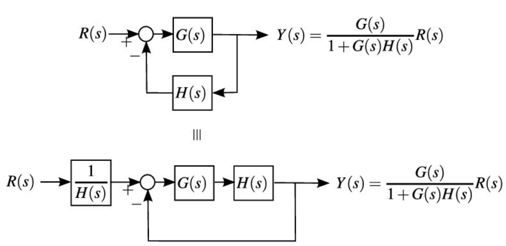
Block diagrams algebra



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Mason's gain formula

This is a formula to calculating the transfer function for a block diagram or a signal flow graph.

$$G(s) = \frac{Y(s)}{R(s)} = \frac{\sum_{i=1}^{n} G_k(s)\Delta_k(s)}{\Delta(s)}$$

with

$$\Delta(s) = 1 - \sum_{i=1}^{n} L_i(s) + \sum L_i(s)L_j(s) - \sum L_i(s)L_j(s)L_k(s) + \dots$$

Notes

Mason's gain formula

Where

- $\Delta(s)$ is the determinant of the graph,
- $G(s)$ is the transfer funtion from input $R(s)$ to output $Y(s)$,
- $R(s)$ is the input signal,
- $Y(s)$ is the output signal,
- n is the number of forward paths between $R(s)$ and $Y(s)$,
- $G_k(s)$ is the transfer funtion for kth forward path between $R(s)$ and $Y(s)$,
- $L_i(s)$ is the loop gain of each closed loop in the system,
- $L_i(s)L_j(s)$ is the product of the loop gains of any two non-touching loops,
- $L_i(s)L_j(s)L_k(s)$ is the product of the loop gains of any three pairwise nontouching loops,
- $\Delta_k(s)$ is the the cofactor value of $\Delta(s)$ for the kth forward path, with the loops touching the kth forward path removed.

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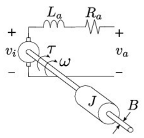
Examples

Notes

Example 1: block diagram for permament magnet DC motor

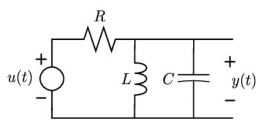


Where:

- v_a : voltage across armature winding (input signal) $[=]V$.
- i_a : current through armature winding $[=]A$.
- R_a : armature winding resistance $[=]\Omega$.
- L_a : armature winding inductance $[=]H$.
- ω : angular velocity in motor axis (output signal) $[=]rad/s$.
- J : moment of inertia of rotating parts of the motor $[=]kg.m^2$.
- B : viscous friction coefficient of rotating parts of the motor $[=]kg.m^2/s$.
- $v_i = K\omega$: voltage induced in the armature $[=]V$. K is a constant associated with the motor.
- $\tau = K\eta i_a$: torque produced by the motor $[=]N \cdot m$. η is the efficiency of the motor $(0 \leq \eta \leq 1)$.

Notes

Example 2: RLC electrical circuit

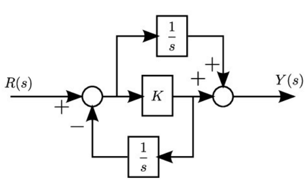


Where:

- $u(t)$: source voltage $[=]V$.
- $y(t)$: output voltage (voltage across L or C) $[=]V$.
- R : electrical resistance $[=]\Omega$.
- L : electrical inductance $[=]H$.
- C : electrical capacitance $[=]F$.

Notes

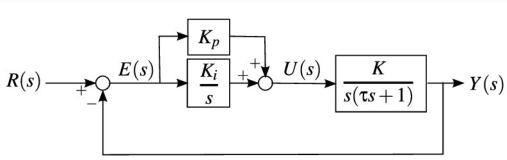
Example 3



Obtain the transfer function of this system: $\dfrac{Y(s)}{R(s)}$.

Notes

Example 4



Obtain the following transfer functions for this system:

1. $\dfrac{Y(s)}{R(s)}$.
2. $\dfrac{U(s)}{R(s)}$.
3. $\dfrac{E(s)}{R(s)}$.

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State space representation of linear time invariant systems

For linear time invariant systems the mathematical model will be of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (1)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (2)$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are constant matrices.

- Note that $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$, $\mathbf{x}(t) \in \mathbb{R}^n$
- Therefore, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$.

Notes

Some forms of the state space representation for linear time invariant systems

- The mathematical representation in state space of a system is not unique given that the state can be represented in infinite ways: one for each base of the state space.
- Let $\mathbf{q}(t)$ be another representation of the state for the same system modeled by equations (1) and (2).
- Let $\mathbf{W} \in \mathbb{R}^{n\times n}$ be the transformation matrix such that

$$\mathbf{x}(t) = \mathbf{W}\mathbf{q}(t) \quad (3)$$

- Note that W must be a invertible matrix.
- Replacing equation (3) in equations (1) and (2),

$$\mathbf{W}\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{W}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{W}\mathbf{q}(t) + \mathbf{D}\mathbf{u}(t)$$

Notes

Some forms of the state space representation for linear time invariant systems

- Therefore,

$$\dot{\mathbf{q}}(t) = \mathbf{W}^{-1}\mathbf{A}\mathbf{W}\mathbf{q}(t) + \mathbf{W}^{-1}\mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{W}\mathbf{q}(t) + \mathbf{D}\mathbf{u}(t)$$

Notes

Some forms of the state space representation for linear time invariant systems

In Summary

$$\dot{\mathbf{q}}(t) = \mathbf{A}'\mathbf{q}(t) + \mathbf{B}'\mathbf{u}(t) \quad (4)$$
$$\mathbf{y}(t) = \mathbf{C}'\mathbf{q}(t) + \mathbf{D}'\mathbf{u}(t) \quad (5)$$

with

$$\begin{aligned} \mathbf{A}' &= \mathbf{W}^{-1}\mathbf{A}\mathbf{W} \\ \mathbf{B}' &= \mathbf{W}^{-1}\mathbf{B} \\ \mathbf{C}' &= \mathbf{C}\mathbf{W} \\ \mathbf{D}' &= \mathbf{D} \end{aligned} \quad (6)$$

Notes

Some forms of the state space representation for linear time invariant systems

Let's discuss three common forms of the state space representation of a linear time invariant system.

- For SISO systems
  - Observable canonical form.
  - Controllable canonical form.
- For any system (SISO/MIMO)
  - The Jordan canonical form.

Notes

Observable canonical form

- Consider a linear time invariant system modeled by the differential equation

$$\left(D^n + a_{n-1}D^{n-1} + \ldots + a_0\right) y(t) = \left(b_n D^n + b_{n-1}D^{n-1} + \ldots + b_0\right) u(t), \quad (7)$$

where a_n has been assumed to be one and $m = n$ without loss of generality.

- The transfer function is given by

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} \quad (8)$$

Notes

Observable canonical form

- Solving for $s^n Y(s)$

$$s^n Y(s) = b_n s^n U(s) - a_{n-1}s^{n-1}Y(s) + b_{n-1}s^{n-1}U(s) + \ldots - a_0 Y(s) + b_0 U(s)$$

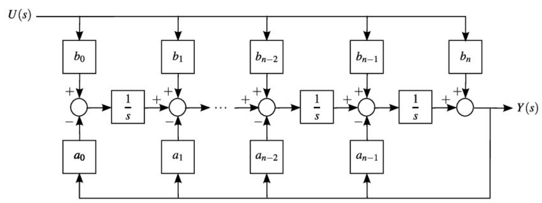
- Integrating n times (multiplying by s^{-n}),

$$Y(s) = b_n U(s) + \frac{1}{s}\left(-a_{n-1}Y(s) + b_{n-1}U(s) + \frac{1}{s}\left(\ldots + \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)\right)\right) \quad (9)$$

Notes

Observable canonical form

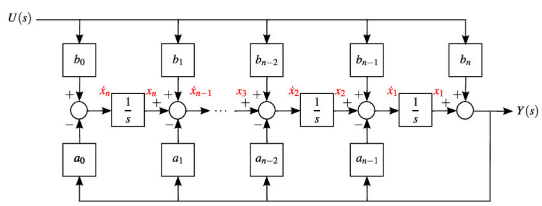
- Equation (9) can be represented by the block diagram



Notes

Observable canonical form

- Defining the state variables as shown



Notes

Observable canonical form

- The state and output equations are obtained

$$
\begin{aligned}
\dot{x}_1(t) &= -a_{n-1}x_1(t) + x_2 + (b_{n-1} - a_{n-1}b_n)\,u(t)\\
\dot{x}_2(t) &= -a_{n-2}x_1(t) + x_3 + (b_{n-2} - a_{n-2}b_n)\,u(t)\\
&\vdots\\
\dot{x}_{n-1}(t) &= -a_1x_1(t) + x_n + (b_1 - a_1b_n)\,u(t)\\
\dot{x}_n(t) &= -a_0x_1(t) + (b_0 - a_0b_n)\,u(t)\\
y(t) &= x_1(t) + b_nu(t)
\end{aligned}
$$

Notes

Observable canonical form

- So the observable canonical for is given by the state and output equations

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\
\mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)
\end{aligned}
$$

- with

$$
\mathbf{A} = \begin{bmatrix} -a_{n-1} & 1 & 0 & & 0\\ -a_{n-2} & 0 & 1 & \dots & 0\\ -a_{n-3} & 0 & 0 & & 0\\ & \vdots & & & \\ -a_1 & 0 & 0 & \dots & 1\\ -a_0 & 0 & 0 & & 0 \end{bmatrix}
\qquad
\mathbf{B} = \begin{bmatrix} b_{n-1} - a_{n-1}b_n\\ b_{n-2} - a_{n-2}b_n\\ b_{n-3} - a_{n-3}b_n\\ \vdots\\ b_1 - a_1b_n\\ b_0 - a_0b_n \end{bmatrix} \tag{10}
$$

$$
\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}
\qquad
\mathbf{D} = \begin{bmatrix} b_n \end{bmatrix}
$$

Notes

Controllable canonical form

- Reconsider a linear time invariant system modeled by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}$$

- This transfer function can be split into two transfer functions connected in series this way

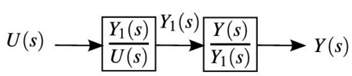
$$\begin{aligned}\frac{Y(s)}{U(s)} &= \frac{Y(s)}{Y_1(s)}\frac{Y_1(s)}{U(s)} \\ &= \left(b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0\right)\frac{1}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}\end{aligned}$$

Notes

Controllable canonical form

So

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{Y_1(s)}\frac{Y_1(s)}{U(s)}$$



where

$$\frac{Y_1(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} \quad (11)$$

$$\frac{Y(s)}{Y_1(s)} = b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0 \quad (12)$$

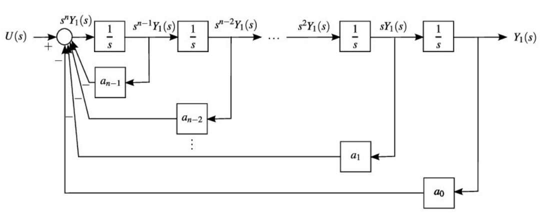
Notes

Controllable canonical form

- Equation (11) can be rewritten as

$$s^n Y_1(s) = -a_{n-1}s^{n-1}Y_1(s) - a_{n-2}s^{n-2}Y_1(s) - \ldots - a_0 Y_1(s) + U(s).$$

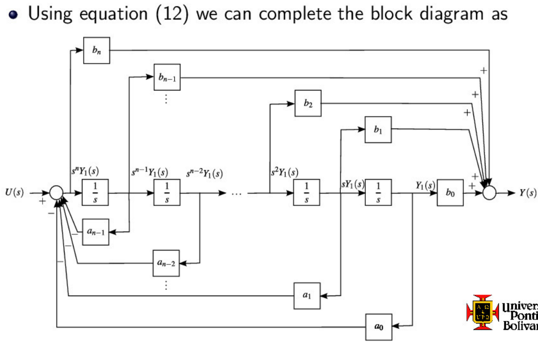
- Which leads to the block diagram



Notes

Controllable canonical form

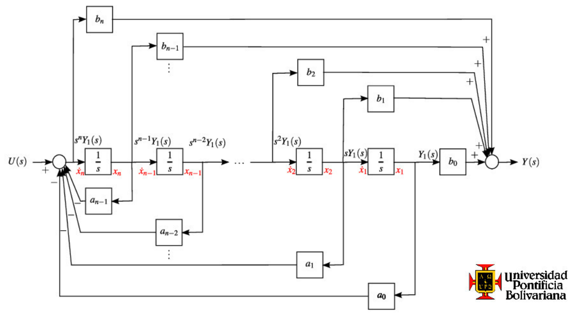
- Using equation (12) we can complete the block diagram as



Notes

Controllable canonical form

- Defining the state variables as shown



Notes

Controllable canonical form

- The state and output equations are obtained

$$\begin{aligned}
\dot{x}_1(t) =& x_2 \\
\dot{x}_2(t) =& x_3 \\
&\vdots \\
\dot{x}_{n-1}(t) =& x_n \\
\dot{x}_n(t) =& -a_0 x_1(t) - a_1 x_2(t) - \ldots - a_{n-1} x_n(t) + u(t) \\
y(t) =& (b_0 - a_0 b_n)\, x_1(t) + (b_1 - a_1 b_n)\, x_2(t) + \ldots + (b_{n-1} - a_{n-1} b_n)\, x_n(t)
\end{aligned}$$

Notes

Controllable canonical form

- So the controllable canonical form is given by the state and output equations

$$\begin{aligned}
\dot{\mathbf{x}}(t) =& \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\
\mathbf{y}(t) =& \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)
\end{aligned}$$

- with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & & 0 \\ & \vdots & & & \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & & -a_{n-1} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{13}$$

$$\mathbf{C} = \begin{bmatrix} b_0 - a_0 b_n & b_1 - a_1 b_n & b_2 - a_2 b_n & \ldots & b_{n-1} - a_{n-1} b_n \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} b_n \end{bmatrix}$$

Notes

Jordan canonical form

- The matrix of transformation $\mathbf{W}$ can be selected such that the new state equations in terms of the new state vector, $\mathbf{q}(t) = \mathbf{W}^{-1}\mathbf{x}(t)$, are uncoupled, that is the new matrix $\mathbf{A}' = \mathbf{W}^{-1}\mathbf{A}\mathbf{W}$ be diagonal.
- Such matrix is the matrix whose columns are the eigenvectors of matrix $\mathbf{A}$.
- That is, if $\mathbf{v_i}$ are the eigenvectors and λ_i are the eigenvalues of matrix $\mathbf{A}$, such that

$$\mathbf{A}\mathbf{v_i} = \lambda_i \mathbf{v_i}, \text{ for } i = 1, \ldots, n$$

- **Assuming the case in which all the eigenvalues are different**, it can be shown that the eigenvectors are linearly independent.
- In this case the matrix of transformation will be $\mathbf{W} = \mathbf{P} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \ldots & \mathbf{v_n} \end{bmatrix}$

Notes

Jordan canonical form

- The Jordan canonical form is the form of the state equations in which the state variables are uncoupled so the matrix of the system is diagonal.
- This form is obtained by transforming any form of the states equations with the transformation matrix $\mathbf{P} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \dots & \mathbf{v_n} \end{bmatrix}$.
- In the Jordan canonical form the state and output equations are

$$\dot{\mathbf{q}}(t) = \mathbf{\Lambda q}(t) + \mathbf{B_n u}(t) \quad (14)$$
$$\mathbf{y}(t) = \mathbf{C_n q}(t) + \mathbf{D_n u}(t) \quad (15)$$

with

$$\begin{aligned} \mathbf{\Lambda} &= \mathbf{P}^{-1}\mathbf{AP} = diag\left(\lambda_1, \lambda_2, \dots, \lambda_n\right) \\ \mathbf{B_n} &= \mathbf{P}^{-1}\mathbf{B} \\ \mathbf{C_n} &= \mathbf{CP} \\ \mathbf{D_n} &= \mathbf{D} \end{aligned} \quad (16)$$

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