Automatic Flight Control

Mathematical Modeling of Dynamical Systems

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Outline

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Mathematical modeling of dynamical systems

- A system model is a representation of the behavior of the system
- The physical model establish the phenomena considered to model the system behavior
- The mathematical model is a set of equations that represents the physical model
- The mathematical model relates the **signals** representing physical variables in the physical model
- In the model there are functions that depends on the **parameters** of the system (i.e. the properties of the components of the model)

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Remarks about mathematical models

- Mathematical models are approximations of real systems behavior
- Models are approximate
  - Approximation in the physical model
  - Approximation in the mathematical model of physical phenomena
- There is also uncertainty
  - Due to external perturbations acting on the system
  - Due to unmodeled phenomena
  - In the parameters of the model
- How detailed and precise is a model depends on the application

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Steps to create a mathematical model for a physical system

- Physical model: which physical phenomena are going to be considered?
- Mathematical model:
  - Define the important variables in the system
    - Through variables
    - Across variables
  - Define the signals and parameters to be considered in the system model
  - Write equations modeling behavior of all components of the system

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Steps to create a mathematical model for a physical system

- Mathematical model:
  - Write equations to relate variables of several components based on conservation principles
    - Equilibrium equations (relationships among through variables)
    - Compatibility equations (relationships among across variables)
  - Combine equations and formulate model

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Mathematical models

We will focus in the following ways to write the mathematical models

- Models in terms of differential equations
- Linear models in terms of transfer functions
- Models in state space

There could be other ways to do the modeling depending on the kind of system

Notes

Mathematical models in terms of differential equations

Using the notation $D \triangleq \frac{d}{dt}$ and $\int \triangleq \int_{-\infty}^{t} (\cdot) d\lambda$, the mathematical model of a dynamical system, with input signal $u(t)$ and output signal $y(t)$, is represented by an ordinary differential equation of the form

$$\left(a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \ldots + a_2 D^2 + a_1 D + a_0\right) y(t) = \left(b_m D^m + b_{m-1} D^{m-1} + \ldots + b_1 D + b_0\right) u(t),$$

where $m \leq n$ for causal systems.

Notes

Mathematical models in terms of differential equations

Using the operator notation

$$L_y\left(y(t)\right) = \left(a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \ldots + a_2 D^2 + a_1 D + a_0\right) y(t)$$

and

$$L_u\left(u(t)\right) = \left(b_m D^m + b_{m-1} D^{m-1} + b_{m-2} D^{m-2} + \ldots + b_2 D^2 + b_1 D + b_0\right) u(t),$$

we can write the differential equation as

$$L_y\left(y(t)\right) = L_u\left(u(t)\right) \quad (1)$$

Notes

Mathematical models in terms of differential equations

- The model represented by equation (1) is linear since operators L_y and L_u are linear.
- That is: $L_y\left(c_1 y_1(t) + c_2 y_2(t)\right) = c_1 L_y\left(y_1(t)\right) + c_2 L_y\left(y_2(t)\right)$ for all c_1, c_2, $y_1(t)$, and $y_2(t)$; and $L_u\left(c_1 u_1(t) + c_2 u_2(t)\right) = c_1 L_u\left(u_1(t)\right) + c_2 L_u\left(u_2(t)\right)$ for all c_1, c_2, $u_1(t)$, and $u_2(t)$
- The model represented by equation (1) is invariant since operators L_y and L_u are invariant (respect to time t).
- That is: if $L_y\left(y(t)\right) = \alpha(t)$, then $L_y\left(y(t-t_0)\right) = \alpha(t-t_0)$ for all $y(t)$ and t_0; and if $L_u\left(u(t)\right) = \beta(t)$, then $L_u\left(u(t-t_0)\right) = \beta(t-t_0)$ for all $u(t)$ and t_0

Notes

Mathematical models in terms of the transfer function

- For a linear time invariant system modeled by

$$\left(a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \ldots + a_2 D^2 + a_1 D + a_0\right) y(t) = \left(b_m D^m + b_{m-1} D^{m-1} + \ldots + b_1 D + b_0\right) u(t),$$

- The transfer function is given by

$$\frac{Y(s)}{U(s)} = \frac{\mathscr{L}\{y(t)\}}{\mathscr{L}\{u(t)\}} = \frac{b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \ldots + b_2 s^2 + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_2 s^2 + a_1 s + a_0}$$

assuming null initial conditions.

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Mathematical models in terms of the transfer function

- If we denote the transfer function of the system by $H(s)$ we get

$$H(s) = \frac{Y(s)}{U(s)} = \left.\frac{\mathscr{L}\{y(t)\}}{\mathscr{L}\{u(t)\}}\right|_{Null\ IC} \tag{2}$$

- Therefore, we can write

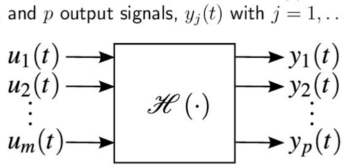
$$Y(s) = \frac{Y(s)}{U(s)} U(s) = H(s) U(s) \tag{3}$$

- That is, **the Laplace transform of the output signal is obtained by multiplying the transfer function of the system times the Laplace transform of the input signal.**

Notes

Mathematical models in state space

Let's consider a system with m input signals, $u_i(t)$ with $i = 1, \ldots, m$, and p output signals, $y_j(t)$ with $j = 1, \ldots, p$.



Notes

Basic concepts about state space

Input vector: is the vector with the input signals: $\mathbf{u}(t) = [u_1(t)\ u_2(t)\ \ldots\ u_m(t)]^T$.

Output vector: is the vector with the output signals: $\mathbf{y}(t) = [y_1(t)\ y_2(t)\ \ldots\ y_p(t)]^T$.

State: is the minimal information required about the system in a time t_0 to be able to determine in a unique maner the outputs of the system, $\mathbf{y}(t)$, for $t \geq t_0$ given that the inputs, $\mathbf{u}(t)$, are known for $t \geq t_0$.

State variables: is a minimal set of variables, $x_i(t)$ with $i = 1, \ldots, n$, used to represent the state of the system.

State vector: is the vector with the state variables: $\mathbf{x}(t) = [x_1(t)\ x_2(t)\ \ldots\ x_n(t)]^T$.

State space: is the set of all possible states of the system. The state space is a vector space of dimension n. n is the order of the system (the minimum number of state variables).

Notes

State space representation of nonlinear time variant systems

- The mathematical model of the system can be represented by

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(t, \mathbf{x}(t), \mathbf{u}(t)\right) \quad (4)$$
$$\mathbf{y}(t) = \mathbf{h}\left(t, \mathbf{x}(t), \mathbf{u}(t)\right) \quad (5)$$

where (4) is the state equation and (5) is the output equation.

- An alternate way of writing equation (4) for some systems is

$$\mathbf{f}\left(t, \dot{\mathbf{x}}(t), \mathbf{x}(t), \mathbf{u}(t)\right) = \mathbf{0} \quad (6)$$

Notes

State space representation of nonlinear time invariant systems

For nonlinear time invariant systems the mathematical model will be of the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right) \quad (7)$$
$$\mathbf{y}(t) = \mathbf{h}\left(\mathbf{x}(t), \mathbf{u}(t)\right) \quad (8)$$

Notes

State space representation of linear time variant systems

For linear time variant systems the mathematical model will be of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \quad (9)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \quad (10)$$

where $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$, and $\mathbf{D}(t)$ are time varying matrices.

- Note that $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$, $\mathbf{x}(t) \in \mathbb{R}^n$
- Therefore $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$, $\mathbf{B}(t) \in \mathbb{R}^{n\times m}$, $\mathbf{C}(t) \in \mathbb{R}^{p\times n}$, and $\mathbf{D}(t) \in \mathbb{R}^{p\times m}$.

Notes

State space representation of linear time invariant systems

For linear time invariant systems the mathematical model will be of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (11)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (12)$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are constant matrices.

- Note that $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$, $\mathbf{x}(t) \in \mathbb{R}^n$
- Therefore, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$.

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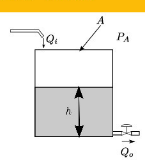
Example 1: cylindrical tank



Where

- Q_i : input flow $[=]m^3/s$.
- Q_o : output flow through output valve $[=]m^3/s$.
- h : level of the tank $[=]m$.
- A : transversal area $[=]m^2$.
- P_A : atmospheric pressure $[=]Pa$.
- ρ : liquid density $[=]kg/m^3$.
- g : gravity acceleration $[=]m/s^2$.

Consider that the valve characteristic is given by $Q_o = K\sqrt{\Delta P}$, where ΔP is the pressure difference across valve and K is a constant associated to the valve.
Find the mathematical model considering Q_i as the input signal and h as the output signal

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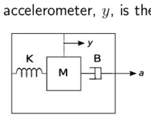
Example 2: accelerometer

Obtain the mathematical model of the accelerometer shown in the figure, where the acceleration of the accelerometer, a, is the input and the displacement of the mass with respect to the body of the accelerometer, y, is the output.



Where

- a : accelerometer acceleration respect to a inertial frame $[=]m/s^2$.
- y : position of the test mass in the accelerometer $[=]m$.
- M : test mass of the accelerometer $[=]kg$.
- K : spring constant $[=]kg/s^2$.
- B : viscous friction constant $[=]kg/s$.

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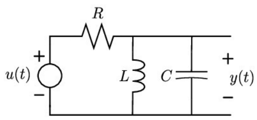
Example 3: RLC electrical circuit

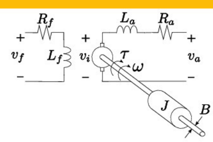


Where:

- $u(t)$: source voltage $[=]V$.
- $y(t)$: output voltage (voltage across L or C) $[=]V$.
- R : electrical resistance $[=]\Omega$.
- L : electrical inductance $[=]H$.
- C : electrical capacitance $[=]F$.

Notes

Example 4: DC motor



Where:

- v_f : voltage applied to field winding $[=]V$.
- i_f : current through field winding $[=]A$.
- v_a : voltage across armature winding $[=]V$.
- i_a : current through armature winding $[=]A$.
- R_f : field winding resistance $[=]\Omega$.
- L_f : field winding inductance $[=]H$.
- R_a : armature winding resistance $[=]\Omega$.
- L_a : armature winding inductance $[=]H$.
- ω : angular velocity in motor axis $[=]rad/s$.
- J : moment of inertia of rotating parts of the motor $[=]kg.m^2$.
- B : viscous friction coefficient of rotating parts of the motor $[=]kg.m^2/s$.
- $v_i = K i_f \omega$: voltage induced in the armature $[=]V$. K is a constant associated with the motor.
- $\tau = K\eta i_f i_a$: torque produced by the motor $[=]N \cdot m$. η is the efficiency of the motor $(0 \leq \eta \leq$

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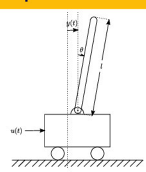
Example 5: inverted pendulum



Where:

- $u(t)$: force applied to car $[=]N$.
- $y(t)$: displacement of car $[=]m$.
- $\theta(t)$: pendulum angle with vertical $[=]rad$.
- M : mass of car $[=]kg$.
- m : mass of pendulum $[=]kg$.
- I : moment of inertia of pendulum $[=]kg \cdot m^2$.
- l : length of pendulum $[=]m$.
- B_c : viscous friction coefficient of car $[=]kg/s$.
- B_p : viscous friction coefficient of pendulum $[=]kg \cdot m^2/s$.

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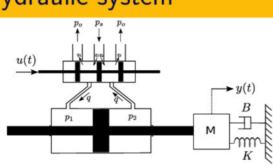
Example 6: hydraulic system



Where:

- $u(t)$: displacement of valve $[=]N$.
- $y(t)$: displacement of mass $[=]m$.
- p_s : supply pressure $[=]Pa$.
- p_o : return pressure $[=]Pa$.
- p_1 : pressure on left side of hydraulic cylinder $[=]Pa$.
- p_2 : pressure on right side of hydraulic cylinder $[=]Pa$.
- q_i : flow through port i $[=]kg/s$.
- q : flow through hydraulic cylinder ports $[=]kg/s$.
- A : hydraulic cylinder area $[=]m^2$.
- ρ : density of hydraulic fluid $[=]kg/m^3$.

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Linearization: the basic concept

- Most mathematical models of real systems are nonlinear
- In many applications of these systems they operate close to some operating condition
- In practice the behavior of these real systems can be approximated by the behavior of linear systems, close to the operating conditions
- This approximation can be done for soft nonlinearities when the deviation from the operating condition is sufficiently small
- This kind of approximation **is not valid for hard nonlinearities** (saturation, deadzone, backslash, hysteresis)

Notes

Linearization: the basic concept

- The basic idea is to approximate nonlinearities by using the first terms of the Taylor series expansion around an operating value: terms of order zero and first order
- Choose the operating condition as an equilibrium point
- Then express the model in terms of the deviations respect to the equilibrium point
- With this procedure the model relating the deviations from the operating point will be linear
- This model is **the linearized model of the system** about that equilibrium point

Notes

Nonlinearities depending on one variable

- Let $y = f(x)$ be a nonlinear behavior
- Here f is a nonlinear function, so $f(a_1x_1 + a_2x_2) \neq a_1f(x_1) + a_2f(x_2)$ for some a_1, a_2, x_1, x_2
- We assume that the f is smooth (continuously differentiable up to order n, with n high enough, in the domain of interest)
- In this case we can use the Taylor series expansion for $f(x)$

$$y = f(x) = f(X_o) + \left.\frac{df(x)}{dx}\right|_{x=X_o}(x - X_o) + H.O.T.$$

- Neglecting Higher Order Terms (H.O.T.) we get

$$y = f(x) = f(X_o) + \left.\frac{df(x)}{dx}\right|_{x=X_o}(x - X_o)$$

Notes

Nonlinearities depending on one variable

- Defining the operating condition by $Y_o = f(X_o)$ and the deviations on y and x as $\tilde{y} = y - Y_o$ and $\tilde{x} = x - X_o$, we get
$$\tilde{y} = \left.\frac{df(x)}{dx}\right|_{x=X_o} \tilde{x}$$
- This is the linearized model for the nonlinear system represented by $y = f(x)$

Notes

Nonlinearities depending on multiple variables

- Let $y = f(x_1, x_2)$ be a nonlinear behavior
- Here f is nonlinear in x_1 and/or x_2
- We assume that the f is smooth (continuously differentiable up to order n, with n high enough, in the domain of interest)
- In this case we can use the Taylor series expansion for $f(x_1, x_2)$
$$y = f(x_1, x_2)$$
$$y = f(X_{1,o}, X_{2,o}) + \left.\frac{\partial f(x_1, x_2)}{\partial x_1}\right|_{\substack{x_1 = X_{1,o} \\ x_2 = X_{2,o}}} (x_1 - X_{1,o}) + \left.\frac{\partial f(x_1, x_2)}{\partial x_2}\right|_{\substack{x_1 = X_{1,o} \\ x_2 = X_{2,o}}} (x_2 - X_{2,o}) + H.O.T.$$

Notes

Nonlinearities depending on multiple variables

- Neglecting Higher Order Terms (H.O.T.) we get
$$y = f(X_{1,o}, X_{2,o}) + \left.\frac{\partial f(x_1, x_2)}{\partial x_1}\right|_{\substack{x_1 = X_{1,o} \\ x_2 = X_{2,o}}} (x_1 - X_{1,o}) + \left.\frac{\partial f(x_1, x_2)}{\partial x_2}\right|_{\substack{x_1 = X_{1,o} \\ x_2 = X_{2,o}}} (x_2 - X_{2,o})$$

Notes

Nonlinearities depending on multiple variables

- Defining the operating condition by $Y_o = f(X_{1,o}, X_{2,o})$ and the deviations on y, x_1, and x_2 as $\tilde{y} = y - Y_o$, $\tilde{x}_1 = x_1 - X_{1,o}$, and $\tilde{x}_2 = x_2 - X_{2,o}$, we get
$$\tilde{y} = \left.\frac{\partial f(x_1, x_2)}{\partial x_1}\right|_{\substack{x_1 = X_{1,o} \\ x_2 = X_{2,o}}} \tilde{x}_1 + \left.\frac{\partial f(x_1, x_2)}{\partial x_2}\right|_{\substack{x_1 = X_{1,o} \\ x_2 = X_{2,o}}} \tilde{x}_2$$
- This is the linearized model for the nonlinear system represented by $y = f(x_1, x_2)$

Notes

Procedure to linearize a nonlinear system model

- For any variable $x(t)$ define an operating condition X_o and a deviation $\tilde{x}(t)$ such that
$$\tilde{x}(t) = x(t) - X_o$$
- Replace in the model any variable $x(t)$ by its decomposition $X_o + \tilde{x}(t)$
- Formulate the equations for the operating condition, setting the deviations to zero in the model
- Solve for the operating condition (equilibrium point)
- Approximate nonlinearities by first terms of Taylor series expansion about the operating condition.
- Subtract the operating condition equations from the model to obtain the linearized model in terms of the deviation of the variables

Notes

Linearization of state space models

- In this case the model of a nonlinear time invariant system is

$$\begin{aligned}\dot{\mathbf{x}}(t) =& f(\mathbf{x}(t), \mathbf{u}(t))\\ \mathbf{y}(t) =& h(\mathbf{x}(t), \mathbf{u}(t))\end{aligned}$$

- Where $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ is the input vector, and $\mathbf{y}(t)$ is the output vector

Notes

Linearization of state space models

- The equilibrium point is found by solving

$$\begin{aligned}0 =& f(\mathbf{X_o}, \mathbf{U_o})\\ \mathbf{Y_o} =& h(\mathbf{X_o}, \mathbf{U_o})\end{aligned}$$

- Defining the deviations

$$\begin{aligned}\tilde{\mathbf{x}}(t) =& \mathbf{x}(t) - \mathbf{X_o},\\ \tilde{\mathbf{u}}(t) =& \mathbf{u}(t) - \mathbf{U_o},\\ \tilde{\mathbf{y}}(t) =& \mathbf{y}(t) - \mathbf{Y_o},\end{aligned}$$

Notes

Linearization of state space models

- We can write the linearized model as

$$\dot{\tilde{\mathbf{x}}}(t) = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x = X_o\\ u = U_o}} \tilde{\mathbf{x}}(t) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x = X_o\\ u = U_o}} \tilde{\mathbf{u}}(t)$$
$$\tilde{\mathbf{y}}(t) = \left.\frac{\partial h(x,u)}{\partial x}\right|_{\substack{x = X_o\\ u = U_o}} \tilde{\mathbf{x}}(t) + \left.\frac{\partial h(x,u)}{\partial u}\right|_{\substack{x = X_o\\ u = U_o}} \tilde{\mathbf{u}}(t)$$

- Where $\tilde{\mathbf{x}}(t)$ is the state vector deviation, $\tilde{\mathbf{u}}(t)$ is the input vector deviation, and $\tilde{\mathbf{y}}(t)$ is the output vector deviation

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Linearization of state space models

- Or

$$\dot{\tilde{\mathbf{x}}}(t) = A\tilde{\mathbf{x}}(t) + B\tilde{\mathbf{u}}(t)$$
$$\mathbf{y}(t) = C\tilde{\mathbf{x}}(t) + D\tilde{\mathbf{u}}(t)$$

Notes

Linearization of state space models

- With

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x = X_o \\ u = U_o}}$$
$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x = X_o \\ u = U_o}}$$
$$C = \left.\frac{\partial h(x,u)}{\partial x}\right|_{\substack{x = X_o \\ u = U_o}}$$
$$D = \left.\frac{\partial h(x,u)}{\partial u}\right|_{\substack{x = X_o \\ u = U_o}}$$

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Examples

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