Universidad Pontificia Bolivariana
Facultad de Ingeniería Aeronáutica
Automatic Flight Control
Laplace transform formulas

Laplace transform definition

Direct Laplace transform

$$X(s) = \mathscr{L}\left\{x(t)\right\} = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

Inverse Laplace transform

$$x(t) = \mathscr{L}^{-1}\left\{X(s)\right\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

with $c + jw \in ROC$ for all $w \in \mathcal{R}$, where ROC is the region of convergence of the Laplace transform of $x(t)$, $X(s)$.

Properties of the Laplace transform

$x(t)$	$X(s) = \mathscr{L}\left\{x(t)\right\}$
$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$
$x(t - t_o)$	$e^{-t_os}X(s)$
$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$
$e^{-at}x(t)$	$X(s+a)$
$tx(t)$	$-\frac{dX(s)}{ds}$
$(-t)^n x(t)$	$\frac{d^nX(s)}{ds^n}$
$\frac{x(t)}{t}$	$\int_s^\infty X(u)du$
$\frac{dx(t)}{dt}$	$sX(s) - x(0^-)$
$\frac{d^nx(t)}{dt^n}$	$s^nX(s) - s^{n-1}x(0^-) - s^{n-2}x^{(1)}(0^-) - \cdots - sx^{(n-2)}(0^-) - x^{(n-1)}(0^-)$
$\int_{-\infty}^{t} x(\lambda)\,d\lambda$	$\frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(t)\,dt}{s}$
$x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda$	$X(s)H(s)$

Initial value theorem

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Final value theorem

$$x(\infty) = \lim_{s\to 0} sX(s)$$

Some Laplace transform pairs

$x(t)$	$X(s) = \mathscr{L}\{x(t)\}$
$\delta(t)$	1
$1(t)$	$\frac{1}{s}$
$r(t)$	$\frac{1}{s^2}$
$p(t) = \int_{-\infty}^{t} r(\lambda)\, d\lambda = \begin{cases} \frac{t^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$	$\frac{1}{s^3}$
$t^n 1(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at} 1(t)$	$\frac{1}{s+a}$
$t^n e^{-at} 1(t)$	$\frac{n!}{(s+a)^{n+1}}$
$Cos\,(w_o t)\, 1(t)$	$\frac{s}{s^2 + w_o^2}$
$Sin\,(w_o t)\, 1(t)$	$\frac{w_o}{s^2 + w_o^2}$
$e^{-at} Cos\,(w_o t)\, 1(t)$	$\frac{s+a}{(s+a)^2 + w_o^2}$
$e^{-at} Sin\,(w_o t)\, 1(t)$	$\frac{w_o}{(s+a)^2 + w_o^2}$
$te^{-at} Cos\,(w_o t)\, 1(t)$	$\frac{(s+a)^2 - w_o^2}{\left[(s+a)^2 + w_o^2\right]^2}$
$te^{-at} Sin\,(w_o t)\, 1(t)$	$\frac{2w_o\,(s+a)}{\left[(s+a)^2 + w_o^2\right]^2}$

The following Laplace transform pairs are valid for $0 \leq \zeta < 1$.

$x(t)$	$X(s) = \mathscr{L}\{x(t)\}$
$\frac{w_n}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} Sin\left(w_n \sqrt{1-\zeta^2}\, t\right) 1(t)$	$\frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$
$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} Sin\left(w_n \sqrt{1-\zeta^2}\, t - Tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right) 1(t)$	$\frac{s}{s^2 + 2\zeta w_n s + w_n^2}$
$\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} Sin\left(w_n \sqrt{1-\zeta^2}\, t + Tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)\right] 1(t)$	$\frac{w_n^2}{s\,(s^2 + 2\zeta w_n s + w_n^2)}$